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editorial

A Big Leap for Man in Astronomy

India even in the ancient days had made large advances in Astronomy. The ancient system is used even today to calculate planetary positions. Sir Issac Newton made further advances. Studies in astronomy in the universities of India are part of Mathematics.

In 1954, when Russia announced that their rocket has successfully planted their country's symbol on the moon, it also started a race for the moon. Finally the astronauts of U.S.A set their foot print on the moon. The scientists of European Space Agency had achieved success beyond our imagination. Their plan is to chase a comet 67P/ Churyumov-Gerasimenko. After a ten year chase, launched in 2004, the probe continues to chase the comet to know more about the composition of the comet. The probe is now 550 million km from the earth.

The comet orbits the Sun once every 6.5 years travelling between the orbits Jupiter and the Earth. The comet is about 4 km wide and the temperature is –70°C. After 10 years, 5 months and 4 days of travelling, looping round the Sun five times Rosetta has caught up with the comet.

The rocket Rosetta and the comet are now 405 million km from the earth. Every information given by the European Space Agency is mind-boggling. This is a victory of man-kind. We wish, the European Space Agency every success.

Anil Ahlawat
Editor

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material. During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 141

JEE MAIN

1. If $ax^3 + bx^2 + c$ is divisible by $x^2 + mx - 1$, then

(a) $a^2 = c(b + c)$ (b) $b^2 = a(c + a)$

(c) $c^2 = b(a + b)$ (d) $a^2 + c(b + c) = 0$

2. There are 10 straight lines in a plane, no two are parallel and no three are concurrent. Their points of intersections are joined. The number of new lines formed is N, with digit sum is_____.

(a) 7 (b) 8 (c) 9 (d) 11

3. $\int_0^{\pi} \cos 6x \cdot \cos^8 x \; dx =$

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{16}$ (c) $\frac{\pi}{32}$ (d) $\frac{\pi}{64}$

4. The shortest distance between the lines $\vec{r} = s(2\hat{i} + 3\hat{j} + 4\hat{k}) - (\hat{i} + \hat{j} + \hat{k})$ and $\vec{r} = t(3\hat{i} + 4\hat{j} + 5\hat{k}) - \hat{i}$ is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{\frac{2}{3}}$ (d) $\frac{1}{\sqrt{6}}$

5. The product of the digits of a 4-digit number is 72. The number of such numbers is N, with digit sum

(a) 4 (b) 5 (c) 6 (d) 10

JEE ADVANCED

6. If 13^{99} is divided by 81, the remainder is divisible by

(a) 3 (b) 5 (c) 7 (d) 11

COMPREHENSION

AOB is a circular sector with bounding radii $OA = OB = 1$. M is the midpoint of the circular arc AB.

Points D and E lie on OA and OB. $\angle AOB = \frac{2\pi}{3}$

7. If triangle MDE is equilateral, its area is

(a) $\frac{\sqrt{3}}{4}$ (b) $\frac{3}{16}$ (c) $\frac{3\sqrt{3}}{16}$ (d) $\frac{\sqrt{3}}{8}$

8. If the triangle MDE is isosceles, its maximum area is

(a) $\frac{\sqrt{3}}{4}$ (b) $\frac{3}{16}$ (c) $\frac{3\sqrt{3}}{16}$ (d) $\frac{\sqrt{3}}{8}$

INTEGER MATCH

9. A game uses a deck of n different cards, $n \geq 6$. If the number of possible sets of 6 cards that can be drawn from the deck is 6 times the number of possible sets of 3 cards that can be drawn. The sum of the digits of n is

MATCHING LIST

10. $S = \{0, 1, 2, 3, 4\}$ and $T = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Column-I		Column-II	
P.	The number of decreasing functions from S to T has digit sum	1.	7
Q.	The number of onto functions from T to S which are non increasing has digit sum	2.	8
R.	The number of onto functions on S such that $f(x) \neq x$, $\forall\ x \in S$ has digit sum	3.	9
S.	The number of non decreasing functions from T to S has digit sum	4.	13

	P	Q	R	S
(a)	4	3	2	1
(b)	1	3	2	4
(c)	3	1	2	4
(d)	1	2	3	4

See Solution set of Maths Musing 140 on page no. 75.

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

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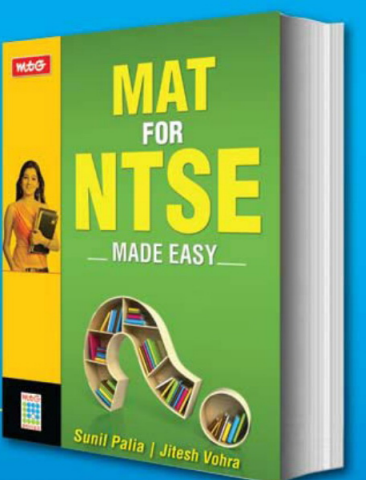

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Challenging Problems For Entrance Exams

SINGLE OPTION CORRECT TYPE

1. $\int \frac{2+x^{-2}}{(x^{-1/2}+x^{3/2})^2}\,dx =$

(a) $\log(x^{-1/2}+x^{3/2})+C$

(b) $\frac{1}{2}\left[\log\left(\frac{x^2}{x^2+1}\right)-\frac{1}{x^2+1}\right]+C$

(c) $\frac{1}{2}\log(x^2+1)-\frac{1}{x^2+1}+C$

(d) None of these

2. $\int \frac{\cot^2 x+1}{\cot^2 x}\log\left(\frac{1+2\tan^2 x}{1+\tan^2 x}\right)dx =$

(a) $\tan x\log(1+\sin^2 x)-2x+\sqrt{2}\tan^{-1}(\sqrt{2}\tan x)+C$

(b) $\log(1+\sin^2 x)-2x+C$

(c) $\tan x\log(1+\sin^2 x)-2x+\sqrt{2}\tan^{-1}(\sqrt{2\tan x})+C$

(d) None of these

3. $\int \frac{x^4\,dx}{(x^2+1007^2)(x^2+1008^2)}$

(a) $x+\frac{1007^3}{2015}\tan^{-1}\frac{x}{1007}+\frac{1008^3}{2015}\tan^{-1}\frac{x}{1008}+C$

(b) $x+\frac{1007}{2015}\tan^{-1}\frac{x}{1007}+\frac{1008}{2015}\tan^{-1}\frac{x}{1008}+C$

(c) $x+\frac{1007^3}{2015}\tan^{-1}\frac{x}{1007}-\frac{1008^3}{2015}\tan^{-1}\frac{x}{1008}+C$

(d) None of these

4. If $f(x)=\int f(x)\,dx$, then $\int\left[f''(x)+\{f'(x)\}^2+\{f(x)\}^3\right]dx=0$ will give $\sum_{r=1}^{3}\frac{\{f(x)\}^r}{r}=?$

(a) $f''(x)$ (b) constant

(c) $f^3(x)+c$ (d) None of these

5. Find $\int(S_x-S_{x-1})dx$ if $S_n=\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)}$

(a) $\frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right|+C$ (b) $\frac{1}{8}\log\left|\frac{1-2x}{1+2x}\right|+C$

(c) $\log\left|\frac{2x-1}{2x+1}\right|+C$ (d) None of these

6. $\int\left[\frac{\sin^{-1}x+\tan^{-1}x+\cos^{-1}x}{\pi}\right]\cdot dx=?$

(where $[\cdot]$ denotes greatest integer function)

(a) $\pi x+C$ (b) $x+C$

(c) $2x+C$ (d) None of these

7. $\int\sqrt{2\sin^3 2x}\cdot\operatorname{cosec}^5 x\,dx=\frac{-8}{\alpha}\{f(x)\}^{\alpha}+C$ then $f(x)$ is undefined if $x=$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{8\pi}{7}$

8. If $\int\frac{(x^4-4)dx}{\sqrt{x^8+x^6+4x^4}}=\sqrt{f\left(x-\frac{2}{x}\right)}+C$, then minimum value of $f(x)$ is

(a) 1 (b) -1 (c) 5 (d) None

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9. Let $f(x) = [x] - x$ (where $[\cdot]$ is *g. i.* function) and $g(x) = \lim_{n\to\infty} \frac{\{f(x)\}^{4n}-1}{\{f(x)\}^{4n}+1}$ then $\int \sum_{r=1}^{2014} \{g(x)\}^r \, dx = ?$

(a) 2014 (b) 2015 (c) 0 (d) None

MULTIPLE OPTION CORRECT TYPE

10. $\int \frac{dx}{\sqrt{\left(x-\frac{2014}{x}\right)\left(\frac{2015}{x}-x\right)}} =$

(a) $\sin^{-1}\sqrt{x^2-2014}+C$

(b) $\cos^{-1}\sqrt{x^2-2014}+C$

(c) $\cos^{-1}\sqrt{2015-x^2}+C$

(d) $\sin^{-1}\sqrt{x^2-2015}+C$

11. $\int \left\{\frac{\sec 2x-\tan 2x}{\sec 2x+\tan 2x}+\frac{2\tan x}{1+\tan x}\right\}e^x dx$

(a) $e^x \tan\left(x-\frac{\pi}{4}\right)+C$ (b) $e^x \cot\left(x-\frac{\pi}{4}\right)+C$

(c) $e^x \cot\left(\frac{\pi}{4}-x\right)+C$ (d) $e^x \tan\left(x+\frac{3\pi}{4}\right)+C$

12. If $f(0) = 1$ and $f'(x)$ exists $\forall\, x \in R$ such that $f(x + y + 1) - f(x) - f(y) = 2\sqrt{f(x)f(y)}$, then $\int f(x)dx = ?$

(a) $(x+1)^2 + C$ (b) $x^2 + 2x + C$

(c) $\frac{(x+1)^3}{3}+C$ (d) $\frac{x^3}{3}+x^2+x+C$

13. If $f(x) = ax^3 + bx^2 + cx + d$ such that $f(0) = 1 = f(1) - 1$ and $x = 0$ is a stationary point but not a point of extrema, then $\int \frac{(x^2+x+1)f(x)dx}{x+1} = \frac{x}{\alpha}+\frac{x^3}{\beta}+\frac{x^5}{\delta}+C$ will show that

(a) α, β, δ are in A.P. (b) only β and δ are prime

(c) $\beta^2 + \alpha = 2\delta$ (d) $\alpha < \beta < \delta$

14. If $\int \frac{x^4+1}{1+x^6}dx = \sum_{r=1}^{3} a_r \tan^{-1} x^r + c$, then

(a) $a_2 = \frac{1}{2}$ (b) $a_3 = \frac{1}{3}$

(c) $\sum_{r=1}^{3} r a_r = 2$ (d) $a_2 < a_3 < a_1$

15. $\int \frac{\sqrt{2\cos ec2x}\cdot\tan\left(\frac{\pi}{4}-x\right)dx}{\sqrt{1+\sin x\cos x}} = fog(x)+C,$

then

(a) $f(x) = \sec^{-1}x$, $g(x) = \sin 2x$

(b) $f(x) = 1 + \sin 2x$, $g(x) = \sec^{-1}x$

(c) $f(x) = \sec^{-1}x$, $g(x) = 1 + \sin 2x$

(d) $f(x) = -\text{cosec}^{-1}x$, $g(x) = 1 + \sin 2x$

SOLUTIONS

1. (b): $\int \frac{(2x^2+1)dx}{x\{\sqrt{x}(x^{-1/2}+x^{3/2})\}^2} = \int \frac{2x^2+1}{x(1+x^2)^2}dx$

$= \int \left\{\frac{1}{x}-\frac{x^3}{(x^2+1)^2}\right\}dx = \log|x| - \frac{1}{2}\int \frac{(z-1)dz}{z^2}$

[We put $x^2 + 1 = z$, $xdx = \frac{1}{2}dz$]

$= \log|x| - \frac{1}{2}\log|z| + \frac{1}{2}\left(-\frac{1}{z}\right)+C$

$= \log|x| - \frac{1}{2}\log|x^2+1| - \frac{1}{2(x^2+1)}+C$

$= \frac{1}{2}\left[\log\left(\frac{x^2}{x^2+1}\right)-\frac{1}{x^2+1}\right]+C$

2. (a): $\int \frac{\text{cosec}^2 x}{\cot^2 x}\log\left(\frac{\cos^2 x+2\sin^2 x}{\cos^2 x+\sin^2 x}\right)dx$

$= \int \sec^2 x \cdot \log(1+\sin^2 x)dx$

$= \log(1+\sin^2 x)\cdot(\tan x) - \int \frac{2\sin x\cos x}{1+\sin^2 x}(\tan x)dx$

$= \tan x \log(1+\sin^2 x) - 2\int \frac{\sin^2 x}{1+\sin^2 x}dx$

$= \tan x \log(1+\sin^2 x) - 2\int \frac{(1+\sin^2 x)-1}{1+\sin^2 x}dx$

$= \tan x \log(1+\sin^2 x) - 2x + 2\int \frac{\sec^2 x\,dx}{(1+\tan^2 x)+\tan^2 x}$

$= \tan x \log(1+\sin^2 x) - 2x + 2\int \frac{dz}{2z^2+1}$

[Put $\tan x = z$]

$$= \tan x \log(1+\sin^2 x) - 2x + \int \frac{dz}{z^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \tan x \log(1+\sin^2 x) - 2x + \sqrt{2}\tan^{-1}\left(\sqrt{2}\tan x\right) + C$$

3. (c) :

$$\int \frac{\{(x^2+1007^2)-1007^2\}\{(x^2+1008^2)-1008^2\}}{(x^2+1007^2)(x^2+1008^2)}dx$$

$$= \int dx - 1007^2 \int \frac{dx}{x^2+1007^2} - 1008^2 \int \frac{dx}{x^2+1008^2}$$

$$+(1007\times1008)^2 \int \frac{dx}{(x^2+1007^2)(x^2+1008^2)}$$

$$= x - 1007^2 \cdot \frac{1}{1007}\tan^{-1}\left(\frac{x}{1007}\right) - 1008^2 \left(\frac{1}{1008}\right)\tan^{-1}\left(\frac{x}{1008}\right)$$

$$+\frac{(1007\times1008)^2}{(1008^2-1007^2)} \cdot \int \left(\frac{1}{x^2+1007^2} - \frac{1}{x^2+1008^2}\right)dx$$

$$= x - 1007\tan^{-1}\left(\frac{x}{1007}\right) - 1008\tan^{-1}\left(\frac{x}{1008}\right)$$

$$+\frac{(1007\times1008)^2}{2015} \cdot \left\{\frac{1}{1007}\tan^{-1}\frac{x}{1007} - \frac{1}{1008}\cdot\tan^{-1}\frac{x}{1008}\right\}dx$$

$$= x + \frac{1007^3}{2015}\tan^{-1}\frac{x}{1007} - \frac{1008^3}{2015}\tan^{-1}\frac{x}{1008} + C$$

4. (b) : Since, $f(x) = \int f(x)dx$

$\therefore\ f'(x) = f(x)$ ...(i)

$$\therefore \int [f''(x) + \{f'(x)\}^2 + \{f(x)\}]^3 dx = 0$$

$$\Rightarrow f'(x) + \int f(x)\, f'(x) + \int \{f(x)\}^2 \cdot f(x)dx = 0$$

$$\Rightarrow f'(x) + \frac{\{f(x)\}^2}{2} + \int \{f(x)\}^2 f'(x)\, dx = 0$$

$$\Rightarrow f'(x) + \frac{\{f(x)\}^2}{2} + \frac{\{f(x)\}^3}{3} + C = 0$$

$$\therefore \frac{f(x)}{1} + \frac{\{f(x)\}^2}{2} + \frac{\{f(x)\}^3}{3} = -C$$

which is constant.

5. (a) : Since, $S_n = \sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)}$

$$\therefore\ t_n = \frac{1}{(2n-1)(2n+1)} = \frac{1}{4n^2-1}$$

$$\therefore\ S_x - S_{x-1} = t_x = \frac{1}{4x^2-1}$$

$$\therefore \int (S_x - S_{x-1})dx = \frac{1}{4}\int \frac{dx}{x^2 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{4}\cdot\frac{1}{2\cdot\frac{1}{2}}\log\left|\frac{x-\frac{1}{2}}{x+\frac{1}{2}}\right| + C$$

$$= \frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right| + C$$

6. (d) : Since, $\sin^{-1}x$, $\tan^{-1}x$ and $\cos^{-1}x$ all are simultaneously defined if $-1 \le x \le 1$ and then $\sin^{-1}x + \tan^{-1}x + \cos^{-1}x$

$$= \frac{x}{2} + \tan^{-1}x \in \left[\frac{\pi}{2} - \frac{\pi}{4}, \frac{\pi}{2} + \frac{\pi}{4}\right] \text{ i.e. } \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

$$\therefore \frac{\sin^{-1}x + \tan^{-1}x + \cos^{-1}x}{\pi} \in \left[\frac{1}{4}, \frac{3}{4}\right]$$

$$\Rightarrow \int \left[\frac{\sin^{-1}x + \tan^{-1}x + \cos^{-1}x}{\pi}\right]dx = \int 0 \cdot dx = C \text{ (constant)}$$

7. (c) : $I = \int \sqrt{2\cdot(2\sin x\cos x)^3}\cdot \operatorname{cosec}^5 x\, dx$

$$= 4\int \sqrt{\sin^3 x\cos^3 x\cdot\operatorname{cosec}^6 x}\cdot\operatorname{cosec}^2 x\, dx$$

$$= 4\int \sqrt{(\cot x)^3}\cdot(\operatorname{cosec}^2 x)dx$$

We put $\cot x = z$

$\therefore\ \operatorname{cosec}^2 x\, dx = -dz$

$$= 4\int z^{3/2}\cdot(-dz) = -4\cdot\frac{z^{5/2}}{5/2} + C = -\frac{8}{5}\left(\sqrt{\cot x}\right)^5 + C$$

$$= -\frac{8}{\alpha}\{f(x)\}^{\alpha} + C \text{ (given)}$$

$\Rightarrow\ \alpha = 5$ and $f(x) = \sqrt{\cot x}$

$\therefore$ $f(x)$ is undefined at $x = \frac{2\pi}{3}$ only

8. (d) : $I = \int \frac{(x^4-4)dx}{x^3\sqrt{x^2+1+\frac{4}{x^2}}} = \int \frac{\left(x - \frac{4}{x^3}\right)dx}{\sqrt{x^2+1+\frac{4}{x^2}}}$

$$= \int \frac{z \cdot dz}{z}$$

$$= z + C = \sqrt{x^2 + 1 + \frac{4}{x^2}} + C \quad = \sqrt{\left(x - \frac{2}{x}\right)^2 + 5} + C$$

We put $x^2 + 1 + \frac{4}{x^2} = z^2 \Rightarrow \left(2x - \frac{8}{x^3}\right) dx = 2z\, dz$

$$\therefore \quad \left(x - \frac{4}{x^3}\right) dx = z \cdot dz$$

$$= \sqrt{f\left(x - \frac{2}{x}\right)} + C \quad \text{(given)}$$

$$\Rightarrow \quad f(x) = x^2 + 5 > 5 \quad [\because \ x \neq 0 \ \ \therefore \ \ x^2 > 0]$$

$\therefore$ Minimum value of $f(x)$ can't be determined.

9. (d) : Since, $f(x) = [x] - x = -\{x\} \in (-1, 0]$ as $\{x\} \in [0, 1)$

$\therefore \ \{f(x)\}^{4n} \to 0$ as $n \to \infty$

$$\therefore g(x) = \lim_{n \to \infty} \frac{\{f(x)\}^{4n} - 1}{\{f(x)\}^{4n} + 1} = \frac{0 - 1}{0 + 1} = -1$$

$$\therefore \quad \int \sum_{r=1}^{2014} \{g(x)\}^r \, dx = \int (g(x) + \{g(x)\}^2 + \{g(x)\}^3 + ... + \{g(x)\}^{2014}) dx$$

$$= \int (-1 + 1 - 1 + 1 - ...\text{upto 2014 terms}) dx$$

$$= \int 0 \cdot dx = c (\text{const of integration})$$

$\because$ c is not definite, nothing can be said about the answer.

10. (a, c): $I = \int \frac{x dx}{\sqrt{(x^2 - 2014)(2015 - x^2)}}$

We put $x^2 - 2014 = z^2 \Rightarrow x dx = z dz$

$$= \int \frac{z \cdot dz}{z \cdot \sqrt{2015 - (z^2 + 2014)}} = \int \frac{dz}{\sqrt{1 - z^2}} = \sin^{-1} z + C$$

$$= \sin^{-1} \sqrt{x^2 - 2014} + C = \cos^{-1} \sqrt{2015 - x^2} + C$$

11. (a,c,d) : $\int \left\{ \frac{1 - \sin 2x}{1 + \sin 2x} + \frac{2 \tan x}{1 + \tan x} \right\} e^x dx$

$$= \int \left\{ \left(\frac{\sin x - \cos x}{\sin x + \cos x} \right)^2 + \frac{(\tan x + 1) + (\tan x - 1)}{1 + \tan x} \right\} e^x dx$$

$$= \int \left\{ \tan^2\left(x - \frac{\pi}{4}\right) + 1 + \frac{\tan x - 1}{\tan x + 1} \right\} e^x dx$$

$$= \int \left\{ \sec^2\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right) \right\} e^x dx$$

$$= \int \sec^2\left(x - \frac{\pi}{4}\right) e^x dx + \tan\left(x - \frac{\pi}{4}\right)(e)^x - \int \sec^2\left(x - \frac{\pi}{4}\right) e^x dx$$

$$= e^x \tan\left(x - \frac{\pi}{4}\right) + C \text{ or, } e^x \cot\left(\frac{\pi}{4} - x\right) + C \text{ or, } e^x \tan\left(x + \frac{3\pi}{4}\right) + C$$

12. (c, d) : Since, $f(0) = 1$ ... (1)

$\because \ f(x + y + 1) - f(x) - f(y) = 2\sqrt{f(x) f(y)}$ ...(2)

$\therefore$ On putting $y = 0$ in (2), we get

$$f(x + 1) = f(x) + f(0) + 2\sqrt{f(x) f(0)}$$

$$= f(x) + 1 + 2\sqrt{f(x)} \ [\because f(0) = 1]$$

On putting $x = 0$, $f(1) = f(0) + 1 + 2\sqrt{f(0)} = 4 = 2^2$

On putting

$x = 1$, $f(2) = f(1) + 1 + 2\sqrt{f(1)} = 4 + 1 + 2\sqrt{4} = 9 = 3^2$

$\therefore$ Clearly, $f(0) = 1 = 1^2$, $f(1) = 2^2$, $f(2) = 3^2$, $f(3) = 4^2$, ... indicates that $f(x) = (x + 1)^2$

$$\therefore \int f(x) dx = \int (x + 1)^2 dx = \frac{(x + 1)^3}{3} + C$$

or, $\frac{x^3}{3} + x^2 + x + c'$ (where $C' = C + \frac{1}{3}$)

13. (a, b, c, d) : $f(0) = 1 \Rightarrow d = 1$

$f(1) = 2 \Rightarrow a + b + c + d = 2 \Rightarrow a + b + c = 1$

$f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$

$\because \ x = 0$ is stationary point but not a point of extrema

$\therefore \ f'(0) = 0 = f''(0) \ \Rightarrow c = 0$ and $2b = 0 \Rightarrow b = 0$

$\because \ a + b + c = 1 \ \ \therefore \ a = 1$

Hence, $f(x) = x^3 + 1$

$$\therefore \int \frac{(x^2 + x + 1) f(x)}{x + 1} dx = \int \frac{(x^2 + x + 1)(x^3 + 1)}{(x + 1)} dx$$

$$= \int \left\{ (x^2 + 1)^2 - x^2 \right\} dx = \int (x^4 + x^2 + 1) dx$$

$$= \frac{x^5}{5} + \frac{x^3}{3} + \frac{x}{1} + C \Rightarrow \alpha = 1, \beta = 3, \delta = 5$$

$\therefore$ (a) α, β, δ, are in A.P.
(b) only β and δ are prime
(c) $\beta^2 + \alpha = 9 + 1 = 10 = 2\delta$
(d) $\alpha < \beta < \delta$

$\therefore$ All options are correct.

Contd. on page 18

PRACTICE PAPER 2Q15

Differential Calculus

1. Suppose $f(x)$ is a differentiable function and $\frac{d^n f(x)}{dx^n}$ is defined for all $n \geq 1$. Suppose $\frac{d^3 f(x)}{dx^3} = f(x)$ and let $g(x) = \cos x + f(x)$. The smallest positive integer n such that $\frac{d^n g(x)}{dx^n} = g(x)$ holds is

(a) 3 (b) 4 (c) 7 (d) 12

2. Let $f(x)$ be a differentiable function and G be the graph of $f(x)$. Let $P = (a, f(a))$ be a point on G closest to $(0, 0)$. Then $f(a) f'(a) =$

(a) a (b) $-a$ (c) 1 (d) -1

3. Suppose $f(x) = x^3 + \log_2(x + \sqrt{x^2 + 1})$. For any $a, b \in R$ to satisfy $f(a) + f(b) \geq 0$, the condition $a + b \geq 0$ is

(a) necessary and sufficient
(b) not necessary but sufficient
(c) necessary but not sufficient
(d) neither necessary nor sufficient

4. For each positive integer n, consider the point P, with x-coordinate n, on the curve $y^2 - x^2 = 1$. If d_n represents the shortest distance from point P to the line $y = x$, then $\lim_{n\to\infty} (n \cdot d_n) =$

(a) 1 (b) $2\sqrt{2}$ (c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{1}{2}$

5. Consider the graph of the function $f(x) = x^4 + 4x^3 - 2x^2 - 6x + 6$. The equation of the unique line which is tangent to this graph in two places is

(a) $y = 6x + 3$ (b) $y = 6x - 3$
(c) $y = 6x + 1$ (d) $y = 6x$

6. f is a twice differentiable function satisfying $f(0) = f'(0) = 1$. If $f''(x) \geq -3$, then the minimum possible value of $f(4)$ is

(a) -16 (b) -19 (c) -20 (d) -21

7. Let $f(x) = 3\sin x + 2\cos x + 1$. If real numbers a, b, c are such that $af(x) + bf(x - c) = 1$ holds for any $x \in R$, then $\frac{b\cos c}{a} =$

(a) 1/2 (b) $-1/2$ (c) 1 (d) -1

8. Let $f_k(x) = 0, x < k$ and $= 1, x \geq k$. The 'T' function is defined to be $T_k(x) = \frac{d}{dx}(f_k(x))$. Suppose $\frac{d^2 g(x)}{dx^2} = T_1(x) + T_2(x)$ and $g(0) = g'(0) = 0$, then $g(5) =$

(a) 6 (b) 7 (c) 9 (d) 10

9. The maximum real value of the function $f(x) = \sqrt{7x - x^2} - \sqrt{9x - x^2 - 14}$ for real x is

(a) $\sqrt{2}$ (b) $\sqrt{10}$ (c) $2\sqrt{5}$ (d) $\sqrt{15}$

10. The function $f : R \to R$ satisfies the equation $f(x + 1) f(x) + f(x + 1) + 1 = 0$ for all x. Then

(a) $f(x)$ is continuous
(b) $f(x)$ is not continuous
(c) $f(x)$ is continuous and differentiable
(d) $f(x)$ is continuous but not differentiable

11. Let $f : R \to R$ be a real function. Out of the following statements which one is true?

(a) If f is continuous and range of $f = R$, then f is monotonic.
(b) If f is monotonic and range of $f = R$, then f is continuous.
(c) If f is monotonic and continuous, then range of $f = R$.
(d) None of the above.

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12. Given $\left(1+\frac{1}{n}\right)^{n+a} \le e \le \left(1+\frac{1}{n}\right)^{n+b}$ for every positive integer n, then

(a) $a_{\max} = 1+\frac{1}{\log 2}$ (b) $a_{\max} = \frac{1}{\log 2}-1$

(c) $b_{\min} = 1+\frac{1}{\log 2}$ (d) $b_{\min} = \frac{1}{\log 2}-1$

13. The number of solutions to the equation in x is/are

$$(\sqrt{2+\sqrt{2}})^x + (\sqrt{2-\sqrt{2}})^x = 2^x$$

(a) zero (b) one
(c) two (d) infinite

14. If $a < b < c$, $f'(x)$ is strictly increasing on (a, c) and $f(x)$ is continuous on $[a, c]$, then
(a) $(b-a)f(c) + (c-b)f(a) > (c-a)f(b)$
(b) $(b-a)f(c) + (c-b)f(a) < (c-a)f(b)$
(c) $(b-a)f(a) + (c-b)f(b) > (c-a)f(c)$
(d) $(b-a)f(a) + (c-b)f(b) < (c-a)f(c)$

15. The number of real valued continuous function f (with domain R) such that if x is rational then $f(x)$ is irrational and if x is irrational then $f(x)$ is rational, is/are
(a) zero (b) two (c) four (d) infinite

16. The graphs of $y = \log_a x$ and $y = a^x (a > 1)$ intersect at exactly one point, then $a =$
(a) e (b) $\sqrt{e}$ (c) e^e (d) $e^{1/e}$

17. For $r \in (0, 1)$, let n_r denote the line that is normal to the curve $y = x^r$ at the point $(1, 1)$ and let A_r denote the region in the first quadrant of the XY plane bounded by the x-axis, the curve $y = x^r$ and the line n_r. The value of r that minimizes the area of A_r is
(a) $\frac{1}{2}$ (b) $\sqrt{2}-1$ (c) $\sqrt{3}-1$ (d) $\frac{1}{3}$

18. Let $f(x) = x^6 - 6x^2 + 6x - 7$. It is known that $f(x)$ has three critical points. The equation of the parabola passing through these three critical points is
(a) $y = -4x^2 + 5x - 7$ (b) $y = 4x^2 - 5x + 7$
(c) $y = -4x^2 - 5x + 7$ (d) $y = 4x^2 - 5x - 7$

19. Tangent lines are drawn at the points of inflexion for the function $f(x) = \cos x$ on $[0, 2\pi]$. The lines intersect with the x-axis so as to form a triangle. The area of this triangle is
(a) $\frac{\pi^2}{2}$ (b) $\frac{\pi^2}{4}$ (c) $\frac{\pi^2}{8}$ (d) $\frac{\pi^2}{16}$

20. Let f be a twice differentiable function such that $f''(t) < 0$ for all $t \in R$. If for two real numbers x and y we have, $f'(y) + x < f(y + 1)$, then
(a) $f(y) = x$ (b) $f(y) > x$
(c) $f(y) < x$ (d) $f(y) < 0$

21. Let $N(x)$ denote the number of positive integers less than x that are not divisible by 2 or 3, then

$$\lim_{x\to\infty} \frac{N(x)}{x} =$$

(a) 1/2 (b) 1/3 (c) 2/3 (d) 3/4

22. Let f be a continuous real function such that $f(11) = 10$ and for all $x, f(x)f(f(x)) = 1$, then $f(9) =$

(a) 9 (b) $\frac{1}{9}$ (c) $\frac{10}{9}$ (d) $\frac{9}{10}$

23. The number of decreasing functions $f: R \to R$ such that $f(f(x)) = x + 1$ for all $x \in R$ is
(a) zero (b) one
(c) two (d) infinite

24. Consider function $f: [0, \infty) \to [0, \infty)$ such that
(i) f is multiplicative, *i.e.*, $f(ab) = f(a)f(b)$ and
(ii) $\lim_{x\to\infty} f(x)$ exists, is finite and different from zero, then $f(4) =$
(a) 1 (b) 2 (c) 3 (d) 4

25. For sets A and B, let $f: A \to B$ and $g: B \to A$ be functions such that $f(g(x)) = x$ for each x. State true (T) or false (F) for each of the following statements.
(i) The function f must be one-one.
(ii) The function f must be onto.
(iii) The function g must be one-one.
(iv) The function g must be onto.
(a) FTTF (b) FTFT
(c) TTFF (d) TFTT

26. Let $f: R \to R$. State true (T) or false (F).
(i) If $|f(x) - f(y)| \le 30|x - y|, \forall\ x, y \in R$, then f must be continuous everywhere.
(ii) If $|f(x) - f(y)| \le 30|x - y|, \forall\ x, y \in R$, then f must be differentiable everywhere.

(iii) If $|f(x) - f(y)| \le 21|x - y|^2$, $\forall\ x, y \in R$, then f must be differentiable everywhere.
(iv) If $|f(x) - f(y)| \le 21|x - y|^2$, $\forall\ x, y \in R$, then f must be constant.
(a) TTTT (b) TFTT (c) FTTF (d) TFTF

27. Let $f(x) = \dfrac{x^4}{(x-1)(x-2)...(x-n)}$, $n \in I^+$. Given that x-axis is a horizontal asymptote for the graph of $f(x)$. What can you deduce about the value of n?
(a) $n = 4$ (b) $n > 4$
(c) $n < 4$ (d) data insufficient

28. Let $f_n(x) = (2 + (-2)^n)x^2 + (n + 3)x + n^2$, where $x \in R$ and n is a positive integer. A possible value of n for which $f_n(x)$ has a maximum value as x varies is
(a) 2 (b) 3 (c) 4 (d) 6

29. For $c > 0$, the equation $\sin x = cx$ has exactly five solutions and x_0 is the largest of these five solutions, then
(a) $\tan x_0 = x_0$ (b) $\cot x_0 = x_0$
(c) $\tan x_0 = 2x_0$ (d) $\cot x_0 = 2x_0$

30. Let $f: (a, b) \to R$ be continuously, differentiable function and $\lim\limits_{x \to a^+} f^2(x) = 0$, $\lim\limits_{x \to b^-} f^2(x) = e - 1$, and $2f(x)f'(x) - f^2(x) \ge 1$ for all $x \in (a, b)$, then possible value of $(b - a)$ is
(a) 1 (b) 2 (c) 3 (d) 4

Solutions

1. (d): Notice that $\dfrac{d^4(\cos x)}{dx^4} = \cos x$.
Also, n needs to be a multiple of 3 as well. So, smallest $n = 12$.

2. (b): Distance function $(D(x)) = x^2 + (f(x))^2$
$D'(x) = 2x + 2f(x)f'(x)$
OP is minimum. So, $D'(a) = 0$
$\Rightarrow\ f(a)f'(a) = -a$

3. (a): Notice that the given $f(x)$ is increasing so, if $a + b \ge 0$, then $a \ge -b$
$\Rightarrow\ f(a) \ge f(-b) \ \Rightarrow\ f(a) \ge -f(b)$
[As f is an odd function]
So, $f(a) + f(b) \ge 0$
And also, $f(a) + f(b) \ge 0$
$\Rightarrow\ f(a) \ge -f(b) = f(-b)$
$\Rightarrow\ a \ge -b \ \Rightarrow\ a + b \ge 0$

4. (c): Consider the point $P\left(n, \sqrt{n^2+1}\right)$ on the curve. Let the vertical line $x = n$ and $y = x$ intersect at $Q(n, n)$.
Now, $\Delta ONQ \sim \Delta PMQ$,
$N \to$ foot of $\perp$ from P on x-axis
$M \to$ foot of $\perp$ from P on $y = x$

$$\Rightarrow\ \frac{d_n}{n} = \frac{\sqrt{n^2+1} - n}{n\sqrt{2}}$$

So, $\lim\limits_{n \to \infty} (n \cdot d_n) = \lim\limits_{n \to \infty} \dfrac{1}{\sqrt{2}\left(1 + \sqrt{1 + \dfrac{1}{n^2}}\right)} = \dfrac{1}{2\sqrt{2}}$

5. (b): Let the line be $y = mx + c$, then
$(x^4 + 4x^3 - 2x^2 - 6x + 6) - (mx + c)$
$= (x^2 + ax + b)^2$
Comparing, same degree terms, $a = 2$, $b = -3$
Hence, line equation becomes $y = 6x - 3$

6. (b): By fundamental theorem of calculus, for any $x \in [0, 4]$

$$f'(x) = 1 + \int_0^x f''(t)dt \ge 1 + \int_0^x -3dt = 1 - 3x$$

And again, $f(x) = 1 + \int_0^x f'(t)dt \ge 1 + \int_0^x (1 - 3t)dt$

$$= 1 + x - \frac{3}{2}x^2$$

So, $f(4)_{\text{least}} = 1 + 4 - \dfrac{3}{2}(4)^2 = -19$

7. (d): Notice that $f(x) + f(x - c) = 2$ for $c = \pi$
So, $\dfrac{1}{2}f(x) + \dfrac{1}{2}f(x - c) = 1$

So, comparing, $a = \dfrac{1}{2}$, $b = \dfrac{1}{2}$ and $c = \pi$

8. (b): $g'(x) = \int (T_1(x) + T_2(x))dx = f_1(x) + f_2(x) + c$
and $g'(0) = 0 \ \Rightarrow\ c = 0$
So, $g'(x) = f_1(x) + f_2(x)$
So, $g(5)$ is basically the area of the two rectangles from $x = 1$ to $x = 5$ and $x = 2$ to $x = 5$.
So, $g(5) = 1 \times 4 + 1 \times 3 = 7$ sq. units

9. (b): Simplify $f(x)$ to get, $f(x) = \dfrac{2\sqrt{7 - x}}{\sqrt{x} + \sqrt{x - 2}}$ with domain $[2, 7]$

As x increases from 2 to 7, numerator decreases and denominator increases and so, $f(x)$ decreases.

So, $f(x)_{\max} = f(2) = \sqrt{10}$

10. (b) : Assume f is continuous and notice that $f(x + 1)$ can never be 0. Further by Intermediate value theorem (IVT), $f(x)$ is always +ve or always –ve. But it is clear from the given equation that $f(x)$ and $f(x + 1)$ cannot both be +ve. So, $f(x) < 0$ for all x. Then

$$0 > f(x+1) = \frac{-1}{1+f(x)} \Rightarrow 1 + f(x) > 0$$

So, we have $0 > f(x) > -1$ for all x and $0 < 1 + f(x) < 1$

But $f(x+1) = \frac{-1}{1+f(x)} < -1$, a contradiction

Thus, f cannot be a continuous function.

11. (b) : Taking counter examples. Consider functions

(a) $f(x) = x^3 - x$

(b) True statement (a graph will suffice the argument)

(c) $f(x) = \tan^{-1}x$

12. (b) : Consider the function $f : [1, \infty) \to R$

$$f(x) = \frac{1}{\log\left(1+\frac{1}{x}\right)} - x, \; f(1) = \frac{1}{\log 2} - 1 \; (> 0)$$

Notice that $f(x)$ is an increasing function and $\lim_{x\to\infty} f(x)$ on applying L'Hospital's rule twice gives the value = 1/2.

So, $f(1) \le f(x) < \frac{1}{2}$

Hence, $\left(1+\frac{1}{x}\right)^{x+\frac{1}{\log 2}-1} \le 1 < \log\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}$

So, for $n \in N$,

$$\left(1+\frac{1}{n}\right)^{n+\frac{1}{\log 2}-1} \le e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}$$

So, $a_{\max} = \frac{1}{\log 2} - 1$ and $b_{\min} = \frac{1}{2}$

13. (b) : The equation can be written as

$f(y) = a^y + (1 - a)^y = 1$

where $2y = x$ and $a = \frac{1}{4}(2+\sqrt{2}), \; (a < 1)$

$f(y)$ is a strictly decreasing function and $f(1) = 1$

$\Rightarrow x = 2$ is the only solution possible.

14. (a) : Applying Mean value theorem,

$$f'(\beta) = \frac{f(c)-f(b)}{c-b} \text{ and } f'(\alpha) = \frac{f(b)-f(a)}{b-a}$$

where $b < \beta < c$ where $a < \alpha < b$

As f' is strictly increasing, so $f'(\beta) > f'(\alpha)$

$$\Rightarrow \frac{f(c)-f(b)}{c-b} > \frac{f(b)-f(a)}{b-a}$$

Now, rearrange the terms to get option (a).

15. (a) : We will show by contradiction that no such function exists.

Consider a function $h(x) = f(x) + x$

Note that $\forall \; x \in R$, $h(x)$ is irrational, but $h(x)$ is continuous.

$\Rightarrow h(x) =$ singleton $= \{k\}$

So, $f(x) = k - x$, but $f(-k) = 2k$, a contradiction to given conditon in question.

16. (d) : Note that $y = \log_a x$, $y = a^x$ are inverse functions.

So, when they intersect they do so on line $y = x$.

So, $a^x \log a = \frac{1}{x \log a} = 1$ (equating slopes)

So, $a = e^{1/e}$

17. (b) : Equation of region A_r is $x = y(1 - r) + 1$.

The area of region A_r is

$$A_r = \int_0^1 (r(1-y) + 1 - y^{1/r})dy = \frac{r}{2} + \frac{1}{r+1}$$

So, A_r has minimum at $r = \sqrt{2} - 1$.

18. (a) : Such a parabola intersects $f(x)$ precisely where $f'(x) = 0$.

So, the value of intersection points do not change if we replace $f(x)$ by $f(x) + g(x) f'(x)$ for any $g(x)$.

So, the parabolic curve is $y = -4x^2 + 5x - 7$.

19. (b) : Inflexion points are $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and tangent equations are $y = -x + \frac{\pi}{2}$ and $y = x - \frac{3\pi}{2}$

Area of $\Delta = \frac{1}{2} \times \frac{\pi}{2} \times \pi = \frac{\pi^2}{4}$

20. (b) : By Mean value theorem,

$$f'(c) = \frac{f(y+1)-f(y)}{(y+1)-y}, \; c \in (y, \; y+1)$$

As $f''(t) < 0 \Rightarrow f'(y) > f'(c)$

So, $f'(y) > \dfrac{f(y+1) - f(y)}{1}$

$\Rightarrow x < f(y) \qquad (\because f'(y) + x < f(y+1))$

21. (b) : Between every pair of multiples of 6, there are two numbers that are coprime to 6. Given x, let y be the largest multiple of $6 \le x$. So, the numbers coprime to $6 \le \dfrac{y}{3}$.

Since, $x < y + 6$, there are at maximum two numbers coprime to 6 between x and y.

Hence, $\dfrac{y}{3} \le N(x) \le 2 + \dfrac{y}{3} \Rightarrow \dfrac{(x-6)}{3} \le N(x) \le \dfrac{x}{3} + 2$

Now, use sandwich role to obtain $\lim\limits_{x\to\infty} \dfrac{N(x)}{x} = \dfrac{1}{3}$

22. (b) : $f(f(x)) = \dfrac{1}{f(x)} \Rightarrow f(y) = \dfrac{1}{y}$

when y is in the range of f.

$f(11) = 10 \Rightarrow 10$ is in the range of f

So, $f(10) = 1/10$

So, 10 and 1/10 are in the range of f.

$\Rightarrow$ By IVT, 9 is in the range of f.

$\Rightarrow f(9) = 1/9$.

23. (a) : Suppose such a f exists, then

$f(f(x)) = f(x) + 1$

$\Rightarrow f(x+1) = f(x) + 1 \Rightarrow f(x+1) > f(x)$

$\Rightarrow f$ is not decreasing, a contradiction.

Hence, no such f exists.

24. (a) : Let $\lim\limits_{t\to\infty} f(t) = c \neq 0$,

for any $(a > 0)$,

$$c = \lim_{t\to\infty} f(at) = \lim_{t\to\infty} f(a) f(t) = f(a) \lim_{t\to\infty} f(t)$$

$\Rightarrow c = f(a)\ .\ c$. So, $f(a) = 1$, identically.

25. (a) : If $g(x_1) = g(x_2)$, then $x_1 = f(g(x_1)) = f(g(x_2)) = x_2$, So, g is one-one.

Also, f is onto because each $x \in B$ is in the image of f, $x = f(g(x))$.

26. (b) : For parts (i), (ii), consider a counter example, $f(x) = |x|$ and for parts (iii) and (iv), we have $\left|\dfrac{f(x) - f(a)}{x - a}\right| \le 21|x - a|$ for some arbitrary quantity a.

As $x \to a$, $|f'(a)| \le 0$

$\Rightarrow f'(a) = 0$ (constant function)

27. (b) : Because $\lim\limits_{x\to\pm\infty} f(x) = 0$, the degree of denominator of $f(x)$ must be greater than degree of numerator.

28. (b) : For any n, $f_n(x)$ is a quadratic in x which has a maximum when the leading coefficient is negative, *i.e.*, $2 + (-2)^n$ is negative,

$\Rightarrow n$ = odd no. greater than 1.

29. (a) : $\sin x_0 = cx_0$ and also, $(\sin x)'_{x_0} = (cx)'_{x_0}$

So, $\cos x_0 = c$. Hence, $\tan x_0 = x_0$

30. (a) : Define $g : (a, b) \to R$,

$g(x) = \log(1 + f^2(x)) - x$

$$g'(x) = \frac{2f(x)f'(x) - 1 - f^2(x)}{1 + f^2(x)} \ge 0, \text{ non decreasing}$$

$(\because \ 2f(x)\, f'(x) - f^2(x) \ge 1)$

So, $\lim\limits_{x\to a^+} g(x) \le \lim\limits_{y\to b^-} g(y) \Rightarrow -a \le 1 - b$

So, $b - a \le 1$

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Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. The centre of a square $ABCD$ is at z_0. If A is z_1, then the centroid of the ΔABC is

(a) $2z_0 - i(z_1 - z_0)$ (b) $z_0 + i\left(\frac{z_1 - z_0}{3}\right)$

(c) $\frac{z_0 + iz_1}{3}$ (d) $\frac{2}{3}(z_1 - z_0)$

2. The reflection of the complex number $(2 - i)$ in the straight line $iz = \bar{z}$ is

(a) $4 - 3i$ (b) $3 + 4i$

(c) $2 + i$ (d) $1 - 2i$

3. If sum of the arguments of all the roots of the equation $z^n = k$ (k is real) is odd multiple of π, then it can be true when

(i) n is odd and $k > 0$ (ii) n is odd and $k < 0$

(iii) n is even and $k > 0$ (iv) n is even and $k < 0$

(a) (i) and (iii) are correct

(b) (i) and (iv) are correct

(c) (ii) and (iii) are correct

(d) (ii) and (iv) are correct

4. Let $f : R \to \left(0, \frac{\pi}{2}\right]$ be a function defined by $f(x) = \cot^{-1}(x^2 + 4x + \alpha^2 - \alpha)$, then complete set of values of α for which $f(x)$ is onto, is

(a) $\left[\frac{1-\sqrt{17}}{2}, \frac{1+\sqrt{17}}{2}\right]$

(b) $\frac{1 \pm \sqrt{17}}{2}$

(c) $\left(-\infty, \frac{1-\sqrt{17}}{2}\right] \cup \left[\frac{1+\sqrt{17}}{2}, \infty\right)$

(d) none of these

5. Let $g(x)$ and $f(x)$ be twice differentiable functions in R and $f(2) = 8$, $g(2) = 0$, $f(4) = 10$ and $g(4) = 8$, then

(a) $g'(x) > 4f'(x)\ \forall\ x \in (2, 4)$

(b) $g(x) > f(x)\ \forall\ x \in (2, 4)$

(c) $3g'(x) = 4f'(x)$ for at least one $x \in (2, 4)$

(d) $g'(x) = 4f'(x)$ for at least one $x \in (2, 4)$

6. If A, G, H be respectively the AM, GM and the HM between two positive numbers and if $xA = yG = zH$ where x, y, z are non zero positive quantities, then x, y, z are in

(a) AP (b) GP

(c) HP (d) AGP

7. Given $\int_0^{\pi/2} \frac{dx}{1 + \sin x + \cos x} = \ln 2$, then the value of the definite integral $\int_0^{\pi/2} \frac{\sin x}{1 + \sin x + \cos x}\, dx$ is

(a) $\frac{1}{2}\ln 2$ (b) $\frac{\pi}{2} - \ln 2$

(c) $\frac{\pi}{4} - \frac{1}{2}\ln 2$ (d) $\frac{\pi}{2} + \ln 2$

8. Let $y = \{x\}^{[x]}$ where $\{x\}$ denotes the fractional part of x and $[x]$ denotes greatest integer $\le x$, then $\int_0^3 y\,dx =$

(a) $\frac{11}{6}$ (b) $\frac{2}{3}$

(c) 1 (d) $\frac{5}{6}$

9. A quadratic equation $f(x) = ax^2 + bx + c = 0$ ($a \neq 0$) has positive distinct roots reciprocal of each other. Which one is correct?

(a) $f'(1) = 0$ (b) $af'(1) < 0$

(c) $af'(1) > 0$

(d) nothing can be said about $af'(1)$

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10. Each of 10 passengers board any of the three buses randomly, which had no passenger initially. The probability that each bus has got at least one passenger, is

(a) $\frac{{}^{10}p_3 3^7}{3^{10}}$ (b) $1-\frac{{}^{10}C_3 3^7}{3^{10}}$

(c) $1-\frac{2^{10}}{3^{10}}$ (d) $\frac{3^{10}-3\cdot 2^{10}+3}{3^{10}}$

SOLUTIONS

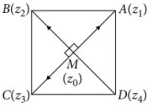
1. (b) : From the figure,

$z_2 - z_0 = i(z_1 - z_0)$

$\Rightarrow z_2 = z_0 + i(z_1 - z_0)$

and $z_3 - z_0 = -(z_1 - z_0)$

$z_3 = 2z_0 - z_1$



Hence, centroid of ΔABC

$= \frac{z_1+z_2+z_3}{3} = z_0 + i\left(\frac{z_1-z_0}{3}\right)$

2. (d) : $iz = \overline{z}$

$\Rightarrow i(x+iy) = x - iy$

$\Rightarrow i(x+y) - (x+y) = 0$

$\Rightarrow (i-1)(x+y) = 0$

which represents the line $y = -x$

So, reflection of the point $(2, -1)$ in the line $y = -x$ gives the point $(1, -2)$.

3. (c): When n is odd and $k > 0$

Sum of arguments = even multiple of π.

When n is odd and $k < 0$

Sum of arguments = odd multiple of π.

When n is even and $k > 0$

Sum of arguments = odd multiple of π.

When n is even and $k < 0$

Sum of arguments = even multiple of π.

4. (b) : Clearly $x^2 + 4x + \alpha^2 - \alpha \geq 0 \ \forall \ x \in R$ and must take all values of the interval $[0, \infty)$

$\Rightarrow D = 0$

i.e. $16 - 4(\alpha^2 - \alpha) = 0 \Rightarrow \alpha^2 - \alpha = 4$

$\Rightarrow \alpha = \frac{1 \pm \sqrt{17}}{2}$.

5. (d) : Let $h(x) = g(x) - 4f(x)$

Verify the Roll's theorem in $(2, 4)$

Now $h(2) = g(2) - 4f(2) = 0 - 4 \times 8 = -32$

$h(4) = g(4) - 4f(4) = 8 - 40 = -32$

$\Rightarrow h'(x) = 0$ for at least one $x \in (2, 4)$

$\therefore \ g'(x) = 4f'(x)$ for at least one $x \in (2, 4)$.

6. (b) : Let two numbers be a and b

$x.\frac{a+b}{2} = y.\sqrt{ab} = z.\frac{2ab}{a+b} \Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{y}{x} = \frac{z}{y}$

$\Rightarrow y^2 = xz \Rightarrow x, y, z$ are in GP.

7. (c): $I = \int_0^{\pi/2} \frac{1+\sin x+\cos x-1-\cos x}{1+\sin x+\cos x}\,dx$

$= \int_0^{\pi/2} dx - \int_0^{\pi/2} \frac{dx}{1+\sin x+\cos x} - \int_0^{\pi/2} \frac{\cos x}{1+\sin x+\cos x}\,dx$

$I = \frac{\pi}{4} - \frac{1}{2}\ln 2$.

8. (a) :

$\int_0^3 (x-[x])^{[x]}dx = \int_0^1 dx + \int_1^2 (x-1)dx + \int_2^3 (x-2)^2 dx$

Put $y = x - 1$, $z = x - 2$

$= 1 + \int_0^1 y\,dy + \int_0^1 z^2\,dz = 1 + \left[\frac{y^2}{2}\right]_0^1 + \left[\frac{z^3}{3}\right]_0^1$

$= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$.

9. (b) : $f(x) = ax^2 + bx + c = 0 \ (a \neq 0)$

Let α and $\frac{1}{\alpha}$ be its roots.

$\alpha + \frac{1}{\alpha} = -\frac{b}{a} > 2$

$\Rightarrow \left(\frac{2a+b}{a}\right) < 0 \Rightarrow a(2a+b) < 0$

$f'(x) = 2ax + b$

$af'(1) = a(2a+b) < 0$

10. (d) : Total ways in which they can be seated in the buses $= 3^{10}$

Number of cases in which each bus has got at least one passenger

$= 3^{10} - {}^3C_1 \times 1 - {}^3C_2(2^{10} - 2)$

$\therefore$ Required probability

$= \frac{3^{10} - {}^3C_1 \times 1 - {}^3C_2(2^{10}-2)}{3^{10}} = \frac{3^{10} - 3\cdot 2^{10} + 3}{3^{10}}$.

TEST YOUR SKILLS

1. Eight people carrying food baskets are going for a picnic on motorcycles. Their names are A, B, C, D, E, F, G and H. They have 4 motorcycles, M1, M2, M3 and M4 among them. They also have 4 food baskets O, P, Q and R of different sizes and shapes and each can be carried only on motorcycles M1, M2, M3 and M4, respectively. No more than 2 persons can travel on a motorcycle and no more than one basket can be carried on a motorcycle. There are 2 husband-wife pairs in this group of 8 people and each pair will ride on a motorcycle together. C cannot travel with A or B. E cannot travel with B or F. G cannot travel with F, H or D. The husband-wife pairs must carry baskets O and P. Q is with A and P is with D. F travels on M1 and E travels on M2 motorcycles. G is with Q, and B cannot go with R. Who is travelling with H?
   (a) A (b) B (c) C (d) D

2. A king has unflinching loyalty from eight of his ministers from M1 to M8, but he has to select only four to make a cabinet committee. He decides to choose these four such that each selected person shares a liking with at least one of the other three selected. The selected person must also aid at least one of the likings of any of the other three person selected.
   - M1 likes fishing and smoking, but hates gambling.
   - M2 likes smoking and drinking, but hates fishing.
   - M3 likes gambling but hates smoking.
   - M4 likes mountaineering but hates gambling.
   - M5 likes drinking but hates smoking and mountaineering.
   - M6 likes fishing but hates smoking and mountaineering.
   - M7 likes gambling and mountaineering but hates fishing.
   - M8 likes smoking and gambling but hates mountaineering.

   Who are four people selected by the king?
   (a) M1, M2, M4 and M6
   (b) M3, M4, M5 and M6
   (c) M4, M5, M6 and M8
   (d) M1, M2, M4 and M7

Directions (3 - 4) : Answer the questions based on the information given below:

Operation on sets are defined as $C = A \cdot B$ implies elements belonging to both A and B. $E = A \cup B$ implies elements belonging to either set A or set B or both. Further a null set (ϕ) is defined as a set that does not contain any element.

In addition, the following sets are defined:
- V is the set of Vertebrates.
- M is the set of Mammals.
- D is the set of Dogs.
- F is the set of Fish.
- A is the set of Alsatians.
- P is a dog named Pluto.

3. $P \cdot A = \phi$ and $P \cup A = D$, then which of the following is true?
   (a) P and A are dogs.
   (b) Pluto is an Alsatian.
   (c) A is not the set of dogs.
   (d) D is a null set.

4. If $Y = F \cup (D \cdot V)$ is not a null set. Then which of the following helds:
   (a) All F are V. (b) All D are V.
   (c) Some F are D. (d) None of these.

Directions (5-6) : Read the statements carefully and choose :

(a) If the question can be solved by any one of the statements but not the other one.
(b) If the question can be solved by using her of the two statements.
(c) If the question can be solved by using both the statements together and not by any one of them.
(d) If the question cannot be solved with the help of given data and more data is required.

5. Ram and Gopal bought apples from a fruit vendor. How many apples were bought by them?
   (i) Ram bought half as many apples as Gopal bought.
   (ii) The dealer had a stock of 500 apples.

6. A boat can ferry 1500 passengers across a river in 12 hours. How many round trips does it make during the journey?
   (i) The boat can carry 200 passengers at a time.
   (ii) During its journey, the boat takes 40 minutes time each way and 20 minutes waiting time at each end.

Directions (7-8) : Refer to the table below and answer the questions that follow.

The chart below represents the statistics pertaining to the employees of Due North Inc.

Length of service (years)	Knowing Shorthand	Knowing Typing	Knowing Both	Total staff
0-5	40/35	35/40	35/30	80/120
6-10	60/15	60/20	40/10	150/75
More than 10	40/40	40/30	15/20	90/110

Following is the representation :
40/35 – Graduate/Non-graduate

7. Out of those employees knowing shorthand only, how many percent are graduates?
(a) 63 (b) 58 (c) 53 (d) 48

8. If 10% of those employees knowing shorthand only learn typing and 12% of those knowing typing only learn shorthand, by what % would the number of persons knowing both increases?
(a) 11 (b) 16 (c) 21 (d) 26

9. On the Island of Who Went Where, there are only two kinds of people. Type No are those who, when they ask a question, must always get a 'No' for an answer and type Yes are those who must always get a 'Yes' for an answer to every question they ask.
Abhay, Lokesh and Rituraj approach you. Abhay asks, "Is it true that neither Lokesh nor Rituraj can be of the type Yes ? " You can infer that
(a) Both Lokesh and Rituraj are No's.
(b) Abhay is a No.
(c) Abhay is a Yes.
(d) None of the above.

10. Six men B, D, C, M, J and K are split in two groups of three each and are made to stand in two rows, such that a man in one row is exactly facing a man in the other row.
M is not at the ends of any row and is to the right of J, who is facing C. K is to the left of D, who is facing M.
Which of the following groups of men are in the same row?
(a) BMD (b) MJK
(c) BDC (d) None of these

ANSWER KEY

1. (c) **2.** (d) **3.** (a) **4.** (a) **5.** (c)
6. (b) **7.** (a) **8.** (a) **9.** (d) **10.** (d)

Challenging Problems For Entrance Exams

Contd. from page 9

14. (b, c, d) :

$$\text{Since, } \int \frac{x^4+1}{1+x^6}dx = \int \frac{(x^4-x^2+1)+x^2}{(x^2+1)(x^4-x^2+1)}dx$$

$$= \int \frac{dx}{1+x^2} + \int \frac{x^2dx}{1+x^6} = \tan^{-1}x + \frac{1}{3}\tan^{-1}x^3 + c$$

$$= \sum_{r=1}^{3} a_r \tan^{-1}x^r + c \text{ (given)}$$

$$\Rightarrow a_1 = 1; a_2 = 0 \text{ and } a_3 = \frac{1}{3} \text{ (on comparison)}$$

$$\therefore \sum_{r=1}^{3} r a_r = 1\cdot a_1 + 2\cdot a_2 + 3\cdot a_3 = 2$$

Also, $a_2 < a_3 < a_1$

15. (c, d) :

$$I = \int \frac{\sqrt{2}}{\sqrt{\sin 2x}} \cdot \frac{\tan\frac{\pi}{4} - \tan x}{1+\tan\frac{\pi}{4}\cdot\tan x} \cdot \frac{dx}{\sqrt{\frac{2+\sin 2x}{2}}}$$

$$= 2\int \frac{\cos x - \sin x}{\cos x + \sin x} \cdot \frac{dx}{\sqrt{\sin 2x}\sqrt{\sin 2x + 2}}$$

$$= 2\int \frac{(\cos^2 x - \sin^2 x)dx}{(\cos x + \sin x)^2\sqrt{(1+\sin 2x)-1}\sqrt{(1+\sin 2x)+1}}$$

$$= 2\int \frac{\cos 2x dx}{(1+\sin 2x)\sqrt{(1+\sin 2x)^2 - 1}}$$

we put $1+\sin 2x = t \quad \therefore 2\cos 2x\, dx = dt$

$$= \int \frac{dt}{t\sqrt{t^2-1}} = \sec^{-1}t + C$$

$$= \begin{cases} \sec^{-1}(1+\sin 2x) + C \\ -\operatorname{cosec}^{-1}(1+\sin 2x) + C' \end{cases} = fog(x) + C \text{ (given)}$$

This is true if $f(x) = \sec^{-1}x$, $g(x) = 1 + \sin 2x$

or,

$f(x) = -\operatorname{cosec}^{-1}x$, $g(x) = 1 + \sin 2x$

$$\left[\because \sec^{-1}x = \frac{\pi}{2} - \operatorname{cosec}^{-1}x \text{ and } C' = C + \frac{\pi}{2}\right]$$

PRACTICE PAPER 2015

JEE (Main & Advanced) and other PETs

STRAIGHT OBJECTIVE TYPE

1. If the roots of $ax^2 + bx + c = 0$ are of the form $\frac{m}{m-1}$ and $\frac{m+1}{m}$, then the value of $(a + b + c)^2$ is

(a) $b^2 - 2ac$ (b) $2b^2 - ac$
(c) $b^2 - 4ac$ (d) $2(b^2 - 2ac)$

2. If the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 3 & 2 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$, then det (A^{-1})=

(a) 1 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 0

3. If two of the roots of the equation $(\lambda - 2)(x^2 + x + 1)^2 - (\lambda + 2)(x^4 + x^2 + 1) = 0$ are equal, then the value (s) of 'λ' is (are) :

(a) $\{-4, 4\}$ (b) $\{-2, 2\}$
(c) $\{2\}$ (d) $\{4\}$

4. A total of 'a' litres of pure acid were drawn from a tank containing 24 litres of pure acid and were replaced by water. The result was thoroughly mixed to obtain a homogeneous solution and then 'a' litres of solution were drawn off, and again replaced by water, and again thoroughly mixed. This procedure was performed 3 times and the tank then contained 3 litres of pure acid. The value of 'a' :

(a) 16 (b) 8 (c) 12 (d) 4

5. The equation $\log_{(2x-3)}(x^2 - 2x) = 1$ has

(a) no solution (b) one solution
(c) two solutions (d) more than 2 solutions

6. The equation $|x - x^2 - 1| = |2x - 4 - x^2|$ has

(a) one solution (b) infinite solutions
(c) two solutions (d) no solution

7. If x denotes the value of the sum of the integers lying between 1 and 100 which are divisible by either 3 or 7. The value of x equals

(a) 5050 (b) 3000
(c) 4068 (d) 2208

8. If p, 1, r be the fourth, seventh and tenth terms (respectively) of a G.P. which of the following is correct?

(a) $p^2 = q^2 + r^2$ (b) $q^2 = pr$
(c) $p^2 = qr$ (d) $pqr + pq + 1 = 0$

9. The sum of the coefficients of the polynomial obtained by collection of the like terms after the expansion of $(1 - 2x + 2x^2)^{743}(2 + 3x - 4x^2)^{744}$ is

(a) 2974 (b) 1487
(c) 1 (d) 0

10. The coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 6 : 33 : 110, then n is :

(a) 10 (b) 11
(c) 12 (d) None of these

11. A biased coin with probability p, $0 < p < 1$ of heads is tossed until a head appears for the first time. If the probability that the number of tosses required is even is 2/5, then p equals

(a) 1/3 (b) 2/3
(c) 2/5 (d) 3/5

12. Out of $(2n + 1)$ tickets numbered consecutively, three are drawn at random. The chance that the numbers on them are in AP is

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(a) $\frac{n}{4n^2-1}$ (b) $\frac{2n}{4n^2-1}$

(c) $\frac{3n}{4n^2-1}$ (d) $\frac{4n}{4n^2-1}$

13. 12 Persons are to be arranged on a round table. If two particular persons among them are not to sit side by side, the number of arrangements is

(a) 9 (10!) (b) 2 (10!)

(c) 45 (8!) (d) 10 !

14. If $[1\ x\ 1]\begin{bmatrix}1 & 3 & 2\\ 0 & 5 & 1\\ 0 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 1\\ x\end{bmatrix} = 0$, then x is equal to

(a) $-3\pm\sqrt{3}$ (b) $\frac{-9\pm\sqrt{53}}{2}$

(c) 1 (d) None of these

15. If $\Delta_a = \begin{vmatrix} a-1 & n & 6\\ (a-1)^2 & 2n^2 & 4n-2\\ (a-1)^3 & 3n^3 & 3n^2-3n\end{vmatrix}$, then $\sum_{a=1}^{n}\Delta_a =$

(a) 0 (b) 1

(c) $\frac{n(n+1)}{2}\cdot\frac{a(a+1)}{2}$ (d) None of these

16. If $\bar{\Delta} = \begin{vmatrix} -1 & 2-3i & 5+4i\\ 2+3i & 8 & 1-i\\ 5-4i & 1+i & 3\end{vmatrix}$, then $\Delta =$

(a) purely real (b) purely imaginary

(c) complex (d) 0

17. If the number $\frac{z-1}{z+1}$ is purely imaginary, then

(a) $|z| = 1$ (b) $|z| > 1$

(c) $|z| < 1$ (d) $|z| > 2$

18. $\sqrt{-1-\sqrt{-1-\sqrt{-1}.....\infty}}$ is equals

(a) ω (b) -1

(c) 0 (d) 1

19. The sum of the infinite terms of the sequence $\frac{5}{3^2\cdot 7^2}+\frac{9}{7^2\cdot 11^2}+\frac{13}{11^2\cdot 15^2}$..... is

(a) $\frac{1}{18}$ (b) $\frac{1}{36}$

(c) $\frac{1}{54}$ (d) $\frac{1}{72}$

20. The number of positive integral solutions of $\frac{x^2(3x-4)^3(x-2)^4}{(x-5)^5(2x-7)^6}\le 0$ is

(a) Four (b) Three

(c) Two (d) Only one

MULTIPLE CORRECT ANSWER TYPE

21. Let $S_n = \frac{3}{2}\cdot\frac{1}{1^2}+\frac{5}{2}\cdot\frac{1+2}{1^2+2^2}+\frac{7}{2}\cdot\frac{1+2+3}{1^2+2^2+3^2}+....$ upto n terms, $n \in N$, then S_n is not greater than

(a) n (b) $2n$

(c) $3n$ (d) 1

22. The value of $\sqrt{i}+\sqrt{-i}$ is/are

(a) $\sqrt{2}$ (b) $-\sqrt{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) $-\frac{1}{\sqrt{2}}$

23. If $\sum_{x=1}^{n}\left(\sum_{m=1}^{k} m^2\right) = an^4+bn^3+cn^2+dn+e$, then

(a) $a=\frac{1}{12}$ (b) $b=\frac{1}{6}$

(c) $d=\frac{1}{6}$ (d) $e=0$

24. If $(1+2x+3x^2)^{10} = a_0+a_1x+a_2x^2+.....+a_{20}x^{20}$, then

(a) $a_1 = 20$ (b) $a_2 = 210$

(c) $a_4 = 8085$ (d) $a_{20} = 2^2\cdot 3^7\cdot 7$

25. In the expansion of $(1+2x-3x^2)^{1025}$, the sum of all odd coefficients is

(a) an even integer (b) an odd integer

(c) greater than 4^{1024} (d) less than 4^{1024}

26. A man takes a step forward with probability 0.4 and backwards with probability 0.6. The probability that at the end of eleven steps he is one step away from the starting point is

(a) ${}^{11}C_6(0.24)^5$ (b) $462\times\left(\frac{16}{25}\right)^5$

(c) $462\times\left(\frac{6}{25}\right)^5$ (d) ${}^{11}C_5(0.24)^5$

27. An urn contains four tickets having numbers 112, 121, 211, 222 written on them. If one ticket is drawn at random and A_i (i = 1, 2, 3) be the events that the i^{th} digit from left of the number on ticket drawn is 1. The events A_1, A_2, A_3 are
(a) pair-wise independent
(b) mutually independent
(c) pair-wise but not mutually independent
(d) mutually but not pair-wise independent

28. If z is a complex number such that $\left|\frac{z-5i}{z+5i}\right| = 1$, then the locus of z is
(a) x-axis
(b) straight line $y = 5$
(c) a circle passing through origin
(d) none of these

29. If z and w are two non-zero examples numbers such that $|zw| = 1$ and Arg (z) – Arg $(w) = \pi/2$, then $\bar{z}\omega$ is equal to
(a) $-i$ (b) 1 (c) -1 (d) i

30. For all $\lambda \in R$, the equation $ax^2 + (b-\lambda)x + (a-b-\lambda) = 0$, $a \neq 0$ has real roots. Then which of the following may be true ?
(a) $a = b$ (b) $b < a < 0$
(c) $b > a > 0$ (d) $a > b > 0$

ANSWER KEY

STRAIGHT OBJECT TYPE

1. (c)	**2.** (c)	**3.** (a)	**4.** (c)
5. (b)	**6.** (a)	**7.** (d)	**8.** (b)
9. (c)	**10.** (c)	**11.** (a)	**12.** (c)
13. (a)	**14.** (b)	**15.** (a)	**16.** (a)
17. (a)	**18.** (a)	**19.** (d)	**20.** (b)

MULTIPLE CORRECT ANSWER TYPE

21. (b),(c) **22.** (a),(b) **23.** (d), (a)
24. (a), (b), (c) **25.** (a), (c) **26.** (a), (c), (d)
27. (a), (c) **28.** (a) **29.** (a)
30. (a), (b), (c)

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CHALLENGING PROBLEMS FOR RMO, InMO, IMO & OTHER CONTESTS

1. Denote by Q^+ the set of all positive rational numbers. Determine all functions $f: Q^+ \to Q^+$ which satisfy the following equation for all $x, y \in Q^+$:

$$f(f(x)^2 y) = x^3 f(xy)$$

2. On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag.

We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

3. Point P lies inside triangle ABC. Lines AP, BP, CP meet the circumcircle of ABC again at points K, L, M, respectively. The tangent to the circumcircle at C meets line AB at S. Prove that $SC = SP$ if and only if $MK = ML$.

4. Find the least positive integer n for which there exists a set $\{s_1, s_2, ..., s_n\}$ consisting of n distinct positive integers such that

$$\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)...\left(1-\frac{1}{s_n}\right)=\frac{51}{2010}.$$

5. Given a positive integer k and other two integers $b > w > 1$. There are two strings of pearls, a string of b black pearls and a string of w white pearls. The length of a string is the number of pearls on it.

On cuts these strings in some steps by the following rules. In each step:

(i) The strings are ordered by their lengths in a non-increasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Then k first ones (if they consist of more than one pearl) are chosen; if there are less than k strings longer than 1, then one chooses all of them.

(ii) Next, one cuts each chosen string into two parts differing in length by at most one. (For instance, if there are strings of 5, 4, 4, 2 black pearls, strings of 8, 4, 3 white pearls and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts (4, 4), (3, 2), (2, 2) and (2, 2), respectively).

The process stops immediately after the step when a first isolated white pearl appears. Prove that at this stage, there will still exist a string of at least two black pearls.

SOLUTIONS

1. We have, $f(f(x)^2 y) = x^3 f(xy)$ ...(i)

By substituting $y = 1$, we get

$f(f(x)^2) = x^3 f(x)$ ...(ii)

Then, whenever $f(x) = f(y)$, we have

$$x^3 = \frac{f(f(x)^2)}{f(x)} = \frac{f(f(y)^2)}{f(y)} = y^3$$

which implies $x = y$, so the function f is injective.

Now, replace x by xy in (ii), and apply (i) twice, second time to $(y, f(x)^2)$ instead of (x, y):

$f(f(xy)^2) = (xy)^3 f(xy)$

$= y^3 f(f(x)^2 y) = f(f(x)^2 f(y)^2)$.

Since f is injective, we get

$f(xy)^2 = f(x)^2 f(y)^2$,

$f(xy) = f(x) f(y)$.

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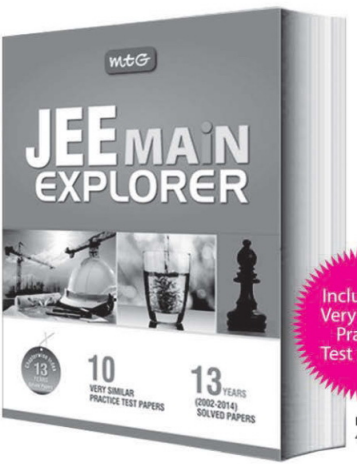

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Therefore, f is multiplicative. This also implies $f(1) = 1$ and $f(x^n) = f(x)^n$ for all integers n.
Then the function equation (i) can be re-written as

$$f(f(x))^2 f(y) = x^3 f(x)\, f(y),$$

$$f(f(x)) = \sqrt{x^3 f(x)} \qquad \text{...(iii)}$$

Let $g(x) = xf(x)$. Then, by (iii), we have

$$g(g(x)) = g(xf(x)) = xf(x)f(xf(x)) = xf(x)^2 f(f(x))$$

$$= xf(x)^2 \sqrt{x^3 f(x)} = (xf(x))^{5/2} = (g(x))^{5/2},$$

and by induction,

$$\underbrace{g(g(\ldots g}_{n+1}(x)\ldots)) = (g(x))^{(5/2)^n} \qquad \text{...(iv)}$$

for every positive integer n.
Consider (iv) for a fixed x. The left-hand side is always rational, so $(g(x))^{(5/2)^n}$ must be rational for every n. We show that this is possible only if $g(x) = 1$. Suppose that $g(x) \neq 1$, and let the prime factorization of $g(x)$ be $g(x) = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$, where $p_1, \ldots, p_k$ are distinct primes and $\alpha_1, \ldots, \alpha_k$ are non-zero integers. Then the unique prime factorization of (iv) is

$$\underbrace{g(g(\ldots g}_{n+1}(x)\ldots)) = (g(x))^{(5/2)^n}$$

$$= p_1^{(5/2)^n \alpha_1} \ldots p_k^{(5/2)^n \alpha_k}$$

where the exponents should be integers. But this is not true for large values of n, for example $\left(\frac{5}{2}\right)^n \alpha_1$ cannot be a integer number when $2^n \nmid \alpha_1$. Therefore, $g(x) \neq 1$ is impossible.

Hence, $g(x) = 1$ and thus $f(x) = \frac{1}{x}$ for all x.

The function $f(x) = \frac{1}{x}$ satisfies the equation (i):

$$f(f(x)^2 y) = \frac{1}{f(x)^2 y} = \frac{1}{\left(\frac{1}{x}\right)^2 y}$$

$$= \frac{x^3}{xy} = x^3 f(xy).$$

Comment : Among $R^+ \to R^+$ functions, $f(x) = \frac{1}{x}$ is not the only solution. Another solution is $f_1(x) = x^{3/2}$. Using transfinite tools infinitely many other solutions can be constructed.

2. When speaking about the diagonal of a square, we will always mean the main diagonal.

Let M_N be the smallest positive integer satisfying the problem condition. First, we show that $M_N > 2^{N-2}$. Consider the collection of all 2^{N-2} flags having yellow first squares and blue second ones. Obviously, both colours appear on the diagonal of each $N \times N$ square formed by these flages.

We are left to show that $M_N \leq 2^{N-2} + 1$, thus obtaining the desired answer. We start with establishing this statement for $N = 4$.

Suppose that we have 5 flags of length 4. We decompose each flag into two parts of 2 squares each; thereby, we denote each flag as LR, where the 2×1 flags L, $R \in S = \{BB, BY, YB, YY\}$ are its left and right parts, respectively. First, we make two easy observations on the flags 2×1 which can be checked manually.

(i) For each $A \in S$, there exists only one 2×1 flag $C \in S$ (possibly $C = A$) such that A and C cannot form a 2×2 square with monochrome diagonal (for part BB, that is YY, and for BY that is YB).

(ii) Let A_1, A_2, $A_3 \in S$ be three distinct elements; then two of them can form a 2×2 square with yellow diagonal, and two of them can form a 2×2 square with blue diagonal (for all parts but BB, a pair (BY, YB) fits for both statements, while for all parts but BY, these pairs are (YB, YY) and (BB, YB)).

Now, let l and r be the numbers of distinct left and right parts of our 5 flags, respectively. The total number of flags is $5 \leq rl$, hence one of the factors (say, r) should be at least 3. On the other hand, $l, r \leq 4$, so there are two flags with coinciding right part; let them be L_1R_1 and L_2R_1 ($L_1 \neq L_2$).

Next, since $r \geq 3$, there exist some flags L_3R_3 and L_4R_4 such that R_1, R_3, R_4 are distinct. Let $L'R'$ be the remaining flag. By (i), one of the pairs (L', L_1) and (L', L_2) can form a 2×2 square with monochrome diagonal; we can assume that L', L_2 form

a square with a blue diagonal. Finally, the right parts of two of the flags L_1R_1, L_3R_3, L_4R_4 can also form a 2×2 square with a blue diagonal by (ii). Putting these 2×2 squares on the diagonal of a 4×4 square, we find a desired arrangement of four flags.

We are ready to prove the problem statement by induction on N; actually, above we have proved the base case $N = 4$. For the induction step, assume that $N > 4$, consider any $2^{N-2} + 1$ flags of length N, and arrange them into a large flag of size $(2^{N-2} + 1) \times N$. This flag contains a non-monochrome column since the flags are distinct; we may assume that this column is the first one. By the pigeonhole principle, this column contains at least

$$\left\lceil \frac{2^{N-2}+1}{2} \right\rceil = 2^{N-3}+1 \text{ squares of one colour}$$

(say, blue). We call the flags with a blue first square good.

Consider all the good flags and remove the first square from each of them. We obtain at least $2^{N-3} + 1 \geq M_{N-1}$ flags of length $N - 1$; by the induction hypothesis, $N - 1$ of them can form a square Q with the monochrome diagonal. Now, returning the removed squares, we obtain a rectangle $(N - 1) \times N$, and our aim is to supplement it on the top by one more flag.

If Q has a yellow diagonal, then we can take each flag with a yellow first square (it exists by a choice of the first column; moreover, it is not used in Q). Conversely, if the diagonal of Q is blue then we can take any of the $\geq 2^{N-3} + 1 - (N - 1) > 0$ remaining good flags. So, in both cases we get a desired $N \times N$ square.

3. We assume that $CA > CB$, so point S lies on the ray AB.

From the similar triangles $\Delta PKM \sim \Delta PCA$ and $\Delta PLM \sim \Delta PCB$ we get $\frac{PM}{KM} = \frac{PA}{CA}$ and $\frac{LM}{PM} = \frac{CB}{PB}$. Multiplying these two equalities, we get

$$\frac{LM}{KM} = \frac{CB}{CA} \cdot \frac{PA}{PB}$$

Hence, the relation $MK = ML$ is equivalent to $\frac{CB}{CA} = \frac{PB}{PA}$.

Denote by E the foot of the bisector of angle B in triangle ABC. Recall that the locus of points X for which $\frac{XA}{XB} = \frac{CA}{CB}$ is the Apollonius circle Ω with the centre Q on the line AB, and this circle passes through C and E. Hence, we have $MK = ML$ if and only if P lies on Ω, that is $QP = QC$.

Now we prove that $S = Q$, thus establishing the problem statement.

We have $\angle CES = \angle CAE + \angle ACE = \angle BCS + \angle ECB = \angle ECS$, so $SC = SE$. Hence, the point S lies on AB as well as on the perpendicular bisector of CE and therefore coincides with Q.

4. Suppose that for some n there exist the desired numbers; we may assume that $s_1 < s_2 < \ldots < s_n$.

Surely $s_1 > 1$ since otherwise $1 - \frac{1}{s_1} = 0$. So we have $2 \leq s_1 \leq s_2 - 1 \leq \ldots \leq s_n - (n - 1)$, hence $s_i \geq i + 1$ for each $i = 1, \ldots, n$. Therefore

$$\frac{51}{2010} = \left(1 - \frac{1}{s_1}\right)\left(1 - \frac{1}{s_2}\right)\ldots\left(1 - \frac{1}{s_n}\right)$$

$$\geq \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\ldots\left(1 - \frac{1}{n+1}\right)$$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n}{n+1} = \frac{1}{n+1},$$

which implies

$$n + 1 \geq \frac{2010}{51} = \frac{670}{17} > 39,$$

so $n \geq 39$.

Now we are left to show that $n = 39$ fits. Consider the set $\{2, 3, \ldots, 33, 35, 36, \ldots, 40, 67\}$ which contains exactly 39 numbers. We have

$$\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{32}{33} \cdot \frac{34}{35} \cdots \frac{39}{40} \cdot \frac{66}{67} = \frac{1}{33} \cdot \frac{34}{40} \cdot \frac{66}{67}$$

$$= \frac{17}{670} = \frac{51}{2010} \qquad \ldots(i)$$

hence for $n = 39$ there exists a desired example.

Comment: One can show that the example (i) is unique.

5. Denote the situation after the i^{th} step by A_i, hence A_0 is the initial situation, and $A_{i-1} \to A_i$ is the i^{th} step. We call a string containing m pearls an m-string; it is an m-w-string or a m-b-string if it is white or black, respectively.

We continue the process until every string consists of a single pearl. We will focus on three moments of the process: (a) the first stage A_s when the first 1-string (no matter black or white) appears; (b) the first stage A_t where the total number of strings is greater than k (if such moment does not appear then we put $t = \infty$); and (c) the first stage A_f when all black pearls are isolated. It is sufficient to prove that in A_{f-1} (or earlier), a 1-w-string appears.

We start with some easy properties of the situations under consideration. Obviously, we have $s \le f$. Moreover, all b-strings from A_{f-1} become single pearls in the f^{th} step, hence all of them are 1-or 2-b-strings.

Next, observe that in each step $A_i \to A_{i+1}$ with $i \le t - 1$, all (> 1)-strings were cut since there are not more than k strings at all; if, in addition, $i < s$, then there were no 1-string, so all the strings were cut in this step.

Now, let B_i and b_i be the lengths of the longest and the shortest b-strings in A_i, and let W_i and w_i be the same for w-strings. We show by induction on $i \le \min\{s, t\}$ that (i) the situation A_i contains exactly 2^i black and 2^i white strings, (ii) $B_i \ge W_i$, and (iii) $b_i \ge w_i$. The base case $i = 0$ is obvious. For the induction step, if $i \le \min\{s, t\}$ then in the i^{th} step, each string is cut, thus the claim (i) follows from the induction hypothesis; next, we have $B_i = [B_{i-1}/2] \ge [W_{i-1}/2] = W_i$ and $b_i = [b_{i-1}/2] \ge [w_{i-1}/2] = w_i$, thus establishing (ii) and (iii).

For the numbers s, t, f, two cases are possible.

Case 1. Suppose that $s \le t$ or $f \le t + 1$ (and hence $s \le t + 1$); in particular, this is true when $t = \infty$. Then in A_{s-1} we have $B_{s-1} \ge W_{s-1}$, $b_{s-1} \ge w_{s-1} > 1$ as $s - 1 \le \min\{s, t\}$. Now, if $s = f$, then in A_{s-1}, there is no 1-w-string as well as no (> 2)-b-string. That is, $2 = B_{s-1} \ge W_{s-1} \ge b_{s-1} \ge w_{s-1} > 1$, hence all these numbers equal 2. This means that in A_{s-1}, all strings contain 2 pearls, and there are 2^{s-1} black and 2^{s-1} white strings, which means $b = 2 \cdot 2^{s-1} = w$. This contradicts the problem conditions.

Hence we have $s \le f - 1$ and thus $s \le t$. Therefore, in the s^{th} step each string is cut into two parts. Now, if a 1-b-string appears in this step, then from $w_{s-1} \le b_{s-1}$ we see that a 1-w-string appears as well; so, in each case in the sth step a 1-w-string appears, while not all black pearls become single, as desired.

Case 2. Now assume that

$$t + 1 \le s \text{ and } t + 2 \le f.$$

Then in A_t we have exactly 2^t white and 2^t black strings, all being larger than 1, and $2^{t+1} > k \ge 2^t$ (the latter holds since 2^t is the total number of strings in A_{t-1}). Now, in the $(t + 1)^{\text{st}}$ step, exactly k strings are cut, not more than 2^t of them being black; so the number of w-strings in A_{t+1} is at least $2^t + (k - 2^t) = k$. Since the number of w-strings does not decrease in our process, in A_{f-1} we have at least k white strings as well.

Finally, in A_{f-1}, all b-strings are not larger than 2, and at least one 2-b-string is cut in the f^{th} step. Therefore, at most $k - 1$ white strings are cut in this step, hence there exists a w-string W which is not cut in the f^{th} step. On the other hand, since a 2-b-string is cut, all (≥ 2)-w-strings should also be cut in the f^{th} step; hence W should be a single pearl. This is exactly what we needed.

Comment: In this solution, we used the condition $b \ne w$ only to avoid the case $b = w = 2^t$. Hence, if a number $b = w$ is not a power of 2, then the problem statement is also valid.

■■

concept BOOSTERS

Class XI

Complex Numbers

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This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

DEFINITION

Complex numbers are defined as expressions of the form $x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$. It is denoted by z, *i.e.* $z = x + iy$. 'x' is called as real part of z (Re z) and 'y' is called as imaginary part of z (Im z).

Every complex number can be regarded as

Purely real	Purely imaginary	Imaginary
if $y = 0$	if $x = 0$	if $y \neq 0$

REMARKS

- $\sqrt{x}\sqrt{y} = \sqrt{xy}$ only if atleast one of either x or y is non-negative.
- $i = \sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.
- Zero is both purely real as well as purely imaginary but not imaginary.
- The set R of real numbers is a proper subset of the set C of complex numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.

CONJUGATE COMPLEX

If $z = x + iy$ be a complex number, then its conjugate complex is obtained by changing the sign of its imaginary part and is denoted by $\overline{z}$, *i.e.* $\overline{z} = x - iy$.

Result :

- $z + \overline{z} = 2\,\mathrm{Re}(z)$
- $z - \overline{z} = 2i\,\mathrm{Im}(z)$
- $z\overline{z} = x^2 + y^2$, which is real
- If z lies in the 1st quadrant, then $\overline{z}$ lies in the 4th quadrant and $-\overline{z}$ lies in the 2nd quadrant.

ALGEBRAIC OPERATIONS

The algebraic operations on complex numbers are similar to those on real numbers treating i as a polynomial. Inequalities in complex numbers are not defined. There is no validity if we say that complex number is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless .

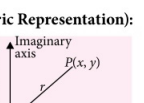
However, in real numbers if $x^2 + y^2 = 0$, then $x = 0 = y$ but in complex numbers, $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

EQUALITY OF COMPLEX NUMBERS

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if their real and imaginary parts coincide.

REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS

(1) Cartesian Form (Geometric Representation):



Every complex number $z = x + iy$ can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).

Length OP is called the modulus of the complex number denoted by $|z|$ and θ is called the argument or amplitude

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i.e., $|z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ (angle made by OP with positive x-axis)

REMARKS

- Argument of a complex number is a many valued function. If θ is the argument of a complex number, then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.
- By specifying the modulus and argument a complex number is defined completely. For the complex number $0 + 0\,i$, the argument is not defined and this is the only complex number which is given by its modulus.
- $|z|$ is always non-negative . Unlike real numbers $|z| = \begin{cases} z, & \text{if } z > 0 \\ -z, & \text{if } z < 0 \end{cases}$ is not correct.
- The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument.
- Unless otherwise stated, amp z implies principal value of the argument.
- There exists a one-one correspondence between the points of the plane and the members of the set of complex numbers.

(2) Trigonometric/Polar Representation

Any complex number in polar form can be represented as

$z = r(\cos\theta + i \sin\theta)$, where $|z| = r$; arg $z = \theta$;

$\bar{z} = r(\cos\theta - i \sin\theta)$

$\cos\theta + i \sin\theta$ is also written as cis θ.

Also, $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2}$

are known as Euler's identities.

(3) Exponential Representation :

Any complex number in exponential form can be written as $z = re^{i\theta}$ where $|z| = r$;

arg $z = \theta$; $\bar{z} = re^{-i\theta}$

IMPORTANT PROPERTIES OF CONJUGATE/ MODULI/AMPLITUDE

If $z, z_1, z_2 \in C$ then

- $z + \bar{z} = 2\,\text{Re}(z)$; $z - \bar{z} = 2i\,\text{Im}(z)$; $\overline{(\bar{z})} = z$;

  $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$; $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$; $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$;

  $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$; $\bar{z}_2 \ne 0$
- $|z| \ge 0$; $|z| \ge \text{Re}(z)$; $|z| \ge \text{Im}(z)$;

  $|z| = |\bar{z}| = |-z|$; $z\bar{z} = |z|^2$; $|z_1 z_2| = |z_1| \cdot |z_2|$;

  $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$; $z_2 \ne 0$, $|z^n| = |z|^n$;

  $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$;

  $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

  Triangle inequality
- (i) $\text{amp}(z_1 \cdot z_2) = \text{amp}\, z_1 + \text{amp}\, z_2 + 2k\pi, k \in I$

  (ii) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp}\, z_1 - \text{amp}\, z_2 + 2k\pi; k \in I$
- $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$,

  where proper value of k must be chosen so that R.H.S. lies in $(-\pi, \pi]$.

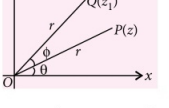
VECTORIAL REPRESENTATION OF A COMPLEX NUMBER

Every complex number can be considered as a point if it is the position vector of that point. If the point P represents the complex number z then, $\overrightarrow{OP} = z$ and $|\overrightarrow{OP}| = |z|$

REMARKS

- If $\overrightarrow{OP} = z = re^{i\theta}$, then $\overrightarrow{OQ} = z_1 = re^{i(\theta+\phi)} = z \cdot e^{i\phi}$

  If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are of unequal magnitude, then $\overrightarrow{OQ} = \overrightarrow{OP}\, e^{i\phi}$.



- If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then

  (a) $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$

  (b) $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$
- If A, B, C and D are four points representing the complex numbers z_1, z_2, z_3 and z_4 respectively then

$AB \parallel CD$, if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely real ;

$AB \perp CD$, if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary

DE-MOIVRE'S THEOREM :

Statement : $\cos n\theta + i\sin n\theta$ is the value or one of the values of $(\cos\theta + i \sin \theta)^n$ for all $n \in Q$. The theorem is very useful in determining the roots of any complex quantity.

Also continued product of the roots of a complex quantity should be determined using theory of equations.

CUBE ROOTS OF UNITY

- The cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$.
- In polar form the cube roots of unity are :

  $\cos 0 + i \sin 0$; $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$;

  $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$
- The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.
- If ω is one of the imaginary cube roots of unity, then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in Z$ but is not the multiple of 3.
- If $a, b, c \in R$ and ω is the cube root of unity, then

  $a^3 - b^3 = (a - b)(a - \omega b)\ (a - \omega^2 b)$;

  $a^2 + a + 1 = (a - \omega)(a - \omega^2)$;

  $a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$;

  $a^3 + b^3 + c^3 - 3abc$

  $= (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

n^{th} ROOTS OF UNITY

If $1, \alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n-1}$ are the n, n^{th} roots of unity, then

- They are in G.P. with common ratio $e^{i(2\pi/n)}$
- $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p$

  $= 0$, if p is not an integral multiple of n

  $= n$ if p is an integral multiple of n
- $(1 - \alpha_1)(1 - \alpha_2)$ ..... $(1 - \alpha_{n-1}) = n$

  $(1 + \alpha_1)(1 + \alpha_2)$ ..... $(1 + \alpha_{n-1})$

  $= 0$, if n is even $= 1$, if n is odd
- $1 \cdot \alpha_1 \cdot \alpha_2 \cdot \alpha_3$ ..... $\alpha_{n-1} = 1$ or -1 according as n is odd or even respectively.

TWO IMPORTANT SERIES

- $\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta$

  $= \frac{\sin(n\theta/2)}{\sin(\theta/2)}\sin\left(\frac{n+1}{2}\right)\theta$
- $\cos\theta + \cos 2\theta + \cos 3\theta + ... + \cos n\theta$

  $= \frac{\sin(n\theta/2)}{\sin(\theta/2)}\cos\left(\frac{n+1}{2}\right)\theta$

Note : If $\theta = (2\pi/n)$, then the sum of the above series vanishes.

STRAIGHT LINES AND CIRCLES IN TERMS OF COMPLEX NUMBERS

If z_1 and z_2 are two complex numbers, then the complex number $z = \frac{nz_1 + mz_2}{m+n}$ divides the line joining the points z_1 and z_2 in the ratio $m : n$.

REMARKS

- Orthocentre of the $\Delta ABC =$

  $$\frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C}$$

  or $\frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$
- If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a, b, c are not all simultaneously zero, then the complex numbers z_1, z_2 and z_3 are collinear.
- The equation of a line joining z_1 and z_2 is given by $z = z_1 + t(z_1 - z_2)$, where t is a parameter.
- If the vertices A, B, C of a triangle represent the complex numbers z_1, z_2, z_3 respectively, then

  (a) Centroid of $\Delta ABC = \frac{z_1 + z_2 + z_3}{3}$

  (b) Incentre of $\Delta ABC = \frac{az_1 + bz_2 + cz_3}{a+b+c}$

  (c) Circumcentre of ΔABC

  $$= \frac{z_1\sin 2A + z_2\sin 2B + z_3\sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

- $\text{amp}(z) = \theta$ is a ray emanating from the origin inclined at an angle θ to the x-axis.
- $|z-a| = |z-b|$ is the equation of perpendicular bisector of the line joining a to b.
- $z = z_1(1+it)$, where t is a real parameter, is a line through the point z_1 and perpendicular to Oz_1.
- The equation of a line passing through z_1 and z_2 can be expressed in the determinant form as $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$. This is also the condition for three complex numbers to be collinear.
- Complex equation of a straight line through two given points z_1 and z_2 can be written as $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + (z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$, which on manipulating takes the form as $\bar{\alpha}z + \alpha\bar{z} + r = 0$, where r is real and α is a non zero complex constant.
- The equation of a circle having centre z_0 and radius ρ is $|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ where r is real, centre $= -\alpha$ and radius $= \sqrt{\alpha\bar{\alpha} - r}$.
  Circle will be real if $\alpha\bar{\alpha} - r \geq 0$.
- The equation of the circle described on the line segment joining z_1 and z_2 as diameter is, $\arg\left(\frac{z-z_1}{z-z_2}\right) = \pm\frac{\pi}{2}$
  or $(z-z_1)(\bar{z}-\bar{z}_2) + (z-z_2)(\bar{z}-\bar{z}_1) = 0$
- Condition for four given points z_1, z_2, z_3 and z_4 to be concyclic is, $\frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1}$ is real.
  Hence, the equation of a circle through three non-collinear points z_1, z_2 and z_3 can be taken as $\frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$ is real
  $\Rightarrow \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)} = \frac{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}$

REFLECTION POINTS FOR A STRAIGHT LINE

- Two given points P and Q are said to be the reflection points for a given straight line if the given line is the perendicular bisector of the segment PQ. Note that the two points denoted by the complex numbers z_1 and z_2 will be the reflection points for the straight line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if; $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$, where r is real and α is non-zero complex constant.
- **Inverse points w.r.t. a circle :** Two points P and Q are said to be inverse w.r.t. a circle with centre 'O' and radius ρ, if,
  (i) the points O, P, Q are collinear and P, Q are on the same side of O.
  (ii) $OP \cdot OQ = \rho^2$.
  Note that the two points z_1 and z_2 will be the inverse points w.r.t. the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if $z_1\bar{z}_2 + \bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$.

PTOLEMY'S THEOREM

It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the lengths of the two pairs of its opposite sides.

i.e. $|z_1 - z_3|\,|z_2 - z_4| = |z_1 - z_2|\,|z_3 - z_4| + |z_1 - z_4|\,|z_2 - z_3|$.

PROBLEMS

SECTION-I

Single Correct Answer Type

1. In a complex plane, the points A and B are at $z_1 = 5 - 2i$ and $z_2 = 1 + i$. If $P(z)$ moves such that $|z - z_1| = 2|z - z_2|$, then the maximum area of ΔPAB is

(a) $\frac{1}{3}$ (b) $\frac{25}{3}$ (c) $\frac{50}{3}$ (d) 5

2. Let z_1, z_2 and z_3 be three points on $|z| = 1$ and $z_1 + z_2 + z_3 = 0$. If θ_1, θ_2 and θ_3 be the arguments of z_1, z_2, z_3 respectively, then $\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_1)$ is equal to

(a) 0 (b) $-\frac{3}{2}$ (c) -1 (d) $\frac{3}{2}$

3. If z and w are two complex numbers simultaneously satisfying the equations, $z^3 + w^5 = 0$

and $z^2 \cdot w^{-4} = 1$, then

(a) z and w both are purely real.

(b) z is purely real and w is purely imaginary.

(c) w is purely real and z is purely imaginary.

(d) z and w both are imaginary.

4. Suppose A is a complex number and $n \in N$, such that $A^n = (A+1)^n = 1$, then the least value of n is

(a) 3 (b) 6 (c) 9 (d) 12

5. If the equation of the perpendicular bisector of the line joining two complex numbers $P(z_1)$ and $Q(z_2)$ in the complex plane is $\bar{\alpha} z + \alpha \bar{z} + r = 0$, then α and r, respectively are

(a) $z_2 - z_1$ and $|z_1|^2 + |z_2|^2$

(b) $z_1 - z_2$ and $|z_1|^2 - |z_2|^2$

(c) $\bar{z}_2 - \bar{z}_1$ and $|z_1|^2 + |z_2|^2$

(d) $z_2 - z_1$ and $|z_1|^2 - |z_2|^2$

6. Given α, β respectively, the fifth and the fourth non-real roots of unity, then the value of $(1+\alpha)(1+\beta)(1+\alpha^2)(1+\beta^2)(1+\beta^3)(1+\alpha^4)$ is

(a) 0

(b) $(1+\alpha+\alpha^2)(1-\beta^2)$

(c) $(1+\alpha)(1+\beta+\beta^2)$ (d) None of these

7. If $z_1 = 2 + 3i$, $z_2 = 3 - 2i$ and $z_3 = -1 - 2\sqrt{3}i$, then which of the following is true?

(a) $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

(b) $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_2}{z_1}\right)$

(c) $\arg\left(\frac{z_3}{z_2}\right) = 2\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

(d) $\arg\left(\frac{z_3}{z_2}\right) = \frac{1}{2}\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

8. Consider the equation $10z^2 - 3iz - k = 0$, where z is a complex variable and $i^2 = -1$. Which of the following statements is true?

(a) For all real positive numbers k, both roots are purely imaginary.

(b) For negative real numbers k, both roots are purely imaginary.

(c) For all pure imaginary numbers k, both roots are real and irrational.

(d) For all complex numbers k, neither root is real.

9. $\sqrt{-1 - \sqrt{-1 - \sqrt{-1 \ldots\ldots \infty}}}$ is equal to

(a) ω or ω^2 (b) $-\omega$ or $-\omega^2$

(c) $1 + i$ or $1 - i$ (d) $-1 + i$ or $-1 - i$

where ω is the imaginary cube root of unity and $i = \sqrt{-1}$

10. $(\sqrt[3]{3} + (3^{5/6})i)^3$ is an integer, where $i = \sqrt{-1}$. The value of the integer is equal to

(a) 24 (b) – 24 (c) – 22 (d) – 21

11. If $x = a + bi$ is a complex number such that $x^2 = 3 + 4i$ and $x^3 = 2 + 11i$ where $i = \sqrt{-1}$, then $(a + b)$ is equal to

(a) 2 (b) 3 (c) 4 (d) 5

12. If $z^3 + (3 + 2i)z + (-1 + ia) = 0$ has one real root, then the value of 'a' lies in the interval ($a \in R$)

(a) $(-2, -1)$ (b) $(-1, 0)$

(c) $(0, 1)$ (d) $(1, 2)$

13. Identify the incorrect statement.

(a) No non-zero complex number z satisfies the equation, $\bar{z} = -4z$

(b) $\bar{z} = z$ implies that z is purely real.

(c) $\bar{z} = -z$ implies that z is purely imaginary.

(d) If z_1, z_2 are the roots of the quadratic equation $az^2 + bz + c = 0$ such that $\mathrm{Im}(z_1 z_2) \neq 0$, then a, b, c must be real numbers .

14. For $z_1 = 6\sqrt{\frac{1-i}{1+i\sqrt{3}}}$; $z_2 = 6\sqrt{\frac{1-i}{\sqrt{3}+i}}$; $z_3 = 6\sqrt{\frac{1+i}{\sqrt{3}-i}}$

which of the following holds good?

(a) $\sum |z_1|^2 = \frac{3}{2}$ (b) $|z_1|^4 + |z_2|^4 = |z_3|^{-8}$

(c) $\sum |z_1|^3 + |z_2|^3 = |z_3|^{-6}$

(d) $|z_1|^4 + |z_2|^4 = |z_3|^8$

15. If α, β be the roots of $ax^2 + bx + c$ ($a, b, c \in R$), $0 < \frac{c}{a} < 1$ and $b^2 - 4ac < 0$, $f(n) = \sum_{r=1}^{n} \left(|\alpha|^r + |\beta|^r \right)$, then $\lim_{n \to \infty} f(n)$ is equal to

(a) $\frac{1}{\sqrt{\frac{a}{c}} - 1}$ (b) $\frac{1}{\sqrt{\frac{a}{c}} + 1}$

(c) $\frac{\sqrt{c}}{-\sqrt{a}+\sqrt{c}}$ (d) $\frac{2}{\sqrt{\frac{a}{c}}-1}$

16. If the argument of $(z-a)(\bar{z}-b)$ is equal to that of $\left(\frac{(\sqrt{3}+i)(1+\sqrt{3}i)}{1+i}\right)$, where a, b are real numbers, if locus of z in the argand diagram is a circle with its centre at $\frac{(3+i)}{2}$, then the ordered pair (a, b) is

(a) (3, 2) (b) (2, 1) (c) (2, 3) (d) (2, 4)

17. If $a = 1+\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^9}{9!}+\ldots$,

$b = x+\frac{x^4}{4!}+\frac{x^7}{7!}+\frac{x^{10}}{10!}+\ldots$ and

$c = \frac{x^2}{2!}+\frac{x^5}{5!}+\frac{x^8}{8!}+\frac{x^{11}}{11!}+\ldots$,

then the value of $a^3 + b^3 + c^3 - 3abc$ is

(a) 0 (b) 1 (c) –1 (d) 1 or –1

18. Let the complex numbers z_1, z_2, z_3 denote the vertices of triangle ABC. Triangles BCD and CAD are equilateral triangle drawn externally to the triangle ABC. If P and Q are the centroids of the two equilateral triangles then $\sum|z_1|^2 + 2\sum \mathrm{Re}(\omega z_1, \bar{z}_2) =$

(a) PQ^2 (b) $2PQ$ (c) $3PQ^2$ (d) 0

19. If for real a, b, c, d, $(a+ib)^{c+id}$ is real, then it is equal to

(a) $(\sqrt{a^2+b^2})^{\sqrt{c^2+d^2}}$ (b) $2^{2\sqrt{(\sqrt{a^2+b^2})^{\sqrt{c^2+d^2}}}}$

(c) $2^{2\sqrt{(a^2+b^2)^{(c^2+d^2)}}}$ (d) $(a^2+b^2)^{2^{2\sqrt{c^2+d^2}}}$

20. If $|z_2 + iz_1| = |z_1| + |z_2|$ and $|z_1| = 3$ and $|z_2| = 4$, then area of ΔABC, if affix of A, B and C, respectively are $(z_1), (z_2)$ and $\left(\frac{z_2 - iz_1}{1-i}\right)$ is

(a) 5/2 sq. units (b) 0 sq. units
(c) 25/2 sq. units (d) 25/4 sq. units

21. If t and c are two complex numbers such that $|t| \neq |c|$, $|t| = 1$ and $z = \frac{at+b}{t-c}$, $z = x + iy$. Locus of z is (where a, b are complex numbers)

(a) line segment (b) straight line
(c) circle (d) none of these

22. The solution of $z^4 + 4z^3i - 6z^2 - 4zi = i$ are the vertices of a convex polygon in the complex plane. The area of the polygon is

(a) $2^{3/4}$ (b) $2^{3/2}$ (c) $2^{5/4}$ (d) 2

23. Let z denotes a complex number and defined $S = \left\{\frac{1}{1-z} : |z| = 1 \text{ and } z \neq 1\right\}$. Which of the following best describes the set S, when S is interpreted geometrically as a set of points in the complex plane?

(a) S is a straight line parallel to the imaginary axis.
(b) S is straight line parallel to the real axis.
(c) S is a circle with a single point missing.
(d) S is a branch of hyperbola.

24. All complex numbers 'z' which satisfies the relation $|z - |z + 1|| = |z + |z - 1||$ on the complex plane lie on the

(a) line $y = 0$
(b) line $x = 0$
(c) circle $x^2 + y^2 = 1$
(d) line $x = 0$ or on a line segment joining $(-1, 0)$ to $(1, 0)$

SECTION-II

More than One Correct Answer Type

25. Suppose that the complex number z lies on the curve such that $\frac{z-4}{z-2i}$ is purely imaginary. If the complex number z_1 represents the mid-point of chord OA of this curve, O being the origin, then z_1 necessarily satisfy

(a) $\frac{z_1-2}{z_1-i} = ik,\ k \in R-\{0\}$

(b) $\frac{z_1}{z_1-2-i} = ik,\ k \in R-\{0\}$

(c) $\frac{z_1-2}{2z_1-i} = k, k \in R-\{0\}$

(d) $|z_1| = \frac{\sqrt{5}}{2}$

26. Let A and B be two distance points denoting the complex numbers α and β respectively. A complex number z lies between A and B, where $z \neq \alpha$, $z \neq \beta$. Which of the following relation(s) hold good?

(a) $|\alpha - z| + |z - \beta| = |\alpha - \beta|$

(b) There exists a positive real number 't' such that $z = (1 - t)\alpha + t\beta$

(c) $\begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0$ (d) $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$

27. Let z_1, z_2, z_3 are three non zero complex numbers denoting the vertices of an equilateral triangle A_1, A_2, A_3 respectively. Consider the statements,

(1) $A_1A_2A_3$ is an equilateral triangle.

(2) $z_1\bar{z}_2 = z_2\bar{z}_3 = z_3\bar{z}_1$

(3) $z_1^2 = z_2z_3$ and $z_2^2 = z_3z_1$

then which of the following is correct?

(a) (2) $\Rightarrow$ (1) (b) (3) $\Rightarrow$ (1)

(c) (2) $\Rightarrow$ (3) (d) (3) $\Rightarrow$ (2)

28. If z_1, z_2, z_3 be three unimodular complex numbers such that $\frac{z_1^2}{z_2z_3} + \frac{z_2^2}{z_1z_3} + \frac{z_3^2}{z_1z_2} + 1 = 0$, then $|z_1 + z_2 + z_3|$ can take the value equal to

(a) 4 (b) 3 (c) 2 (d) 1

29. Let $z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3, z_4, \bar{z}_4, z_5, \bar{z}_5$ be the roots of the equation $x^{10} + (13x - 1)^{10} = 0$. If $N = \sum_{i=1}^{5} \frac{1}{|z_i|^2}$, then

(a) the digit in the units place of N is 0.

(b) the sum of the digits in N is 13.

(c) the digit in the 10's place of N is 5.

(d) the product of digits of N is 0.

30. Let a complex number α, $(\alpha \neq 1)$ to be root of the equation $z^{p+q} - z^p - z^q + 1 = 0$, where p, q are distinct primes. Then

(a) $1 + \alpha + \alpha^2 + ... + \alpha^{p-1} = 0$ and $1 + \alpha + \alpha^2 + ... + \alpha^{q-1} \neq 0$

(b) $1 + \alpha + \alpha^2 + ... + \alpha^{p-1} \neq 0$ and $1 + \alpha + \alpha^2 + ... + \alpha^{q-1} = 0$

(c) $1 + \alpha + \alpha^2 + ... + \alpha^{p-1} = 0$ and $1 + \alpha + \alpha^2 + ... + \alpha^{q-1} = 0$

(d) $1 + \alpha + \alpha^2 + ... + \alpha^{p-1} \neq 0$ and $1 + \alpha + \alpha^2 + ... + \alpha^{q-1} \neq 0$

31. If x and y are real numbers and $\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$, then

(a) $x = 3$ (b) $y = 1$

(c) $y = -1$ (d) $x = -3$

32. If $\arg(z^{3/8}) = (1/2)\arg(z^2 + \bar{z}\, z^{1/2})$, then which of the following is (are) true ?

(a) $|z| = 1$ (b) z is real

(c) z is purely imaginary (d) $z^{1/2} = 1$

33. For any complex number $z = x + iy$, defined by $z = |x| + |y|$. If z_1 and z_2 are any complex numbers, then

(a) $(z_1 + z_2) \leq (z_1) + (z_2)$

(b) $(z_1 + z_2) = (z_1) + (z_2)$

(c) $(z_1 + z_2) \geq (z_1) + (z_2)$

(d) $|(z_1 + z_2)| \leq |(z_1)| + |(z_2)|$

34. Let $z_1 = 1 + i$, $z_2 = -1 - i$ and z_3 be complex numbers such that z_1, z_2 and z_3 form an equilateral triangle. Then z_3 is equal to

(a) $\sqrt{3}(1+i)$ (b) $\sqrt{3}(1-i)$

(c) $\sqrt{3}(i-1)$ (d) $\sqrt{3}(-1-i)$

35. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at M and satisfy $BD = 2AC$. If the points D and M are represented by the complex numbers $1 + i$ and $2 - i$, respectively, then A is represented by

(a) $3 - \frac{i}{2}$ (b) $3 + \frac{i}{2}$

(c) $1 + \frac{3i}{2}$ (d) $1 - \frac{3i}{2}$

36. If for any two complex numbers z_1 and z_2, $(|z_2| \neq 1)$ $\sqrt{z_1} - i\sqrt{z_2} = |z_2|\sqrt{z_1} + i|z_1|\sqrt{z_2}$, then which of the following options are correct?

(a) $z_1\bar{z}_2 + 1 = 0$ (b) $z_1 + z_2 = 0$

(c) $\arg\left(\frac{z_1}{z_2}\right) = \pi$ (d) $\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$

37. If z_1, z_2, z_3, z_4, z_5 and z_6 are the vertices of a regular hexagon in anticlockwise direction. If z_1, z_2 are given, then

(a) $z_3 = -\frac{1}{2}(1 + \sqrt{3}\,i)z_1 + \frac{1}{2}(3 + \sqrt{3}i)z_2$

(b) $z_4 = -\sqrt{3}i\, z_1 + (1 + \sqrt{3}i)z_2$

(c) $z_5 = (1-\sqrt{3}i)z_1 + \sqrt{3}i\, z_2$

(d) $z_6 = \frac{1}{2}(3-\sqrt{3}i)z_1 - \frac{1}{2}(1-\sqrt{3}i)z_2$

38. If $z_1 = 5 + 12i$ and $|z_2| = 4$, then

(a) maximum $(|z_1 + iz_2|) = 17$

(b) minimum $(|z_1 + (1+i)z_2|) = 13 - 4\sqrt{2}$

(c) minimum $\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{4}$

(d) maximum $\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{3}$

39. If the equation $az^2 + z + 1 = 0$ has a purely imaginary root, where $a = \cos\theta + i\sin\theta$, $i = \sqrt{-1}$, then the interval in which the function $f(x) = x^3 - 3x^2 + 3(1 + \cos\theta)x + 5$ is increasing, is/are

(a) $(-\infty, -2)$ (b) $(2, 5)$

(c) $R - (1, 2)$ (d) $(5, \infty)$

SECTION-III

Comprehension Type

Paragraph for Question No. 40 to 42

On the sides AB and BC of ΔABC, squares are drawn with centres D and E such that points C and D lie on the same side of line AB and the points A and E lie on either side of line BC.

A, B, C are represented by the complex numbers 1, ω, ω^2 respectively. $\left(\text{Here } \omega = \frac{-1+i\sqrt{3}}{2}\right)$

40. Angle between AC and DE is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

41. The length of DE is

(a) $\sqrt{\frac{3}{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\sqrt{3}$ (d) $\sqrt{6}$

42. The length of AE is

(a) $\frac{3+\sqrt{3}}{2}$ (b) $\frac{3-\sqrt{3}}{2}$

(c) $\frac{3-\sqrt{3}}{4}$ (d) $\frac{3+\sqrt{3}}{4}$

Paragraph for Question No. 43 to 45

If $7\theta = (2n + 1)\pi$, where $n = 0, 1, 2, 3, 4, 5, 6$ then answer the following questions.

43. The equations whose roots are

$\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$ is

(a) $8x^3 + 4x^2 + 4x + 1 = 0$

(b) $8x^3 - 4x^2 - 4x - 1 = 0$

(c) $8x^3 - 4x^2 - 4x + 1 = 0$

(d) $8x^3 + 4x^2 + 4x - 1 = 0$

44. The value of $\sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7}$ is

(a) 4 (b) −4 (c) 3 (d) −3

45. The value of $\sec^2\frac{\pi}{7} + \sec^2\frac{3\pi}{7} + \sec^2\frac{5\pi}{7}$ is

(a) −24 (b) 80 (c) 24 (d) −80

Paragraph for Question No. 46 to 48

Let A, B, C be three sets of complex numbers as defined below,

$A = \{z : \text{Im } z \geq 1\}$; $B = \{z : |z - 2 - i| = 3\}$

$C = \{z : \text{Re}((1 - i)z) = \sqrt{2}\}$

46. The number of elements in the set $A \cap B \cap C$ is

(a) 0 (b) 1 (c) 2 (d) ∞

47. Let z be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(a) 25 and 29 (b) 30 and 34

(c) 35 and 39 (d) 40 and 44

48. Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between

(a) −6 and 3 (b) −3 and 6

(c) −6 and 6 (d) −3 and 9

Paragraph for Question No. 49 to 51

Points $D(z_1)$, $E(z_2)$, $F(z_3)$ lie on a circle centred at the origin O. The tangents to the circle at D, E, F intersect at A, B and C. A, B, C are opposite to D, E, F respectively.

49. If A is represented by z, then $\arg\left(\frac{z_3 - z}{z_3}\right)$ is equal to

(a) $-\frac{\pi}{2}$ (b) $-\frac{\pi}{3}$ (c) $-\pi$ (d) $-\frac{2\pi}{3}$

50. The complex number z is given by

(a) $\dfrac{2z_1z_3}{z_1+z_3}$ (b) $\dfrac{2z_2z_3}{z_2+z_3}$

(c) $\dfrac{2z_1z_2}{z_1+z_2}$ (d) none of these

51. The complex representation of M (point of intersection of AO and DE) is given by

(a) $\left(\dfrac{z_1+z_3}{z_1+z_2}\right)z_1$ (b) $\left(\dfrac{z_1-z_3}{z_1-z_2}\right)z_1$

(c) $\left(\dfrac{z_1+z_2}{z_1+z_3}\right)z_3$ (d) $\dfrac{z_1z_3}{z_1+z_3}$

Paragraph for Question No. 52 to 54

The complex slopes of a line passing through two points represented by complex numbers z_1 and z_2 is defined by $\dfrac{z_2-z_1}{\bar{z}_2-\bar{z}_1}$ and we shall denote it by ω.

If z_0 is a complex number and c is a real number, then $\bar{z}_0z+z_0\bar{z}+c=0$ represents a straight line. Its complex slope is $-\dfrac{z_0}{\bar{z}_0}$.

Now consider two lines

$$\alpha\bar{z}+\bar{\alpha}z+i\beta=0 \quad \text{....(i)}$$

and $a\bar{z}+\bar{a}z+b=0$ ....(ii)

where α, β and a, b are complex constants and let their complex slopes be denoted by ω_1 and ω_2 respectively.

52. If the lines are inclined at an angle of 120° to each other, then

(a) $\omega_2\bar{\omega}_1=\omega_1\bar{\omega}_1$ (b) $\omega_2\bar{\omega}_1^{\,2}=\omega_1\bar{\omega}_2^{\,2}$

(c) $\omega_1^2=\omega_2^2$ (d) $\omega_1+2\omega_2=0$

53. Which of the following must be true ?

(a) a must be purely imaginary

(b) β must be purely imaginary

(c) a must be real

(d) b must be imaginary

54. If line (i) makes an angle of 45° with real axis, then $(1+i)\left(-\dfrac{2\alpha}{\bar{\alpha}}\right)$ is

(a) $2\sqrt{2}$ (b) $2\sqrt{2}i$

(c) $2(1-i)$ (d) $-2(1+i)$

SECTION-IV

Matrix-Match Type

55.

	Column - I		Column - II
(A)	Let ω be a non real cube root of unity, then the number of distinct elements in the set $\{(1+\omega+\omega^2+\ldots+\omega^n)^m \mid m, n \in N\}$ is	(p)	4
(B)	Let 1, ω, ω^2 be the cube root of unity. The least possible degree of a polynomial with real coefficient having roots 2ω, $(2+3\omega)$, $(2+3\omega^2)$, $(2-\omega-\omega^2)$, $(2-\omega-\omega^2)$, is	(q)	5
(C)	If $\alpha = 6+4i$ and $\beta = 2+4i$ are two complex numbers on the complex plane. A complex number z satisfying $\text{amp}\left(\dfrac{z-\alpha}{z-\beta}\right)=\dfrac{\pi}{6}$ moves on a major segment of a circle whose radius is	(r)	6
		(s)	8

56. The complex numbers $z_1, z_2, z_3, \ldots, z_n$ taken in order represent the consecutive vertices of a regular polygon of n sides and $z_3+z_n=Az_1+\bar{A}z_2$, where A is a complex number. Let [.] denotes the greatest integer function. Then,

	Column - I (n)		Column - II [$\lvert A \rvert$]
(A)	4	(p)	0
(B)	6	(q)	1
(C)	8	(r)	2
(D)	12	(s)	3

57. Match the following :

	Column - I		Column - II
(A)	If z_1, z_2, z_3 are the vertices of an equi-lateral triangle then	(p)	$(z_1-z_2)^2+3(z_3-z_2)^2=0$

(B)	If z_1, z_2, z_3 are vertices of A, B, C respectively of an isosceles right angled triangle with right angle at C, then	(q)	$(z_2 - z_3)^2 = 3(z_1 - z_2)(z_3 - z_1)$
(C)	If z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles triangle and angles B and C are each equal to $\pi/6$, then	(r)	$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$
(D)	If z_1, z_2, z_3 are the vertices A, B, C respectively of a triangle ABC in which $\angle A = \frac{\pi}{6}, \angle B = \frac{\pi}{2}$ and $\angle C = \frac{\pi}{3}$, then	(s)	$(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$

58. Match the following :

Column - I *(Equation of circle)*		Column - II *(Radius)*	
(A)	$\lvert z-1\rvert^2 + \lvert z+1\rvert^2 = 4$	(p)	3
(B)	$\text{Amp}\left(\frac{z-2i}{z+2i}\right) = \frac{\pi}{2}$	(q)	4
(C)	$z\bar{z} - z(1-2i) - \bar{z}(1+2i) = 4$	(r)	2
(D)	$\lvert z+3\rvert = 2\lvert z-3\rvert$	(s)	1

59. Match the following :

Column - I		Column - II	
(A)	Locus of the point 'z' satisfying the equation $\text{Re}(z^2) = \text{Re}(z + \bar{z})$ is	(p)	a straight line
(B)	Locus of the point 'z' satisfying the equation $\lvert z - z_1\rvert + \lvert z - z_2\rvert = \lambda$, $\lambda \in R^+$ and z_1, z_2 are two fixed complex numbers such that $\lambda > \lvert z_1 - z_2\rvert$ is	(q)	a circle
(C)	Locus of the point 'z' satisfying the equation $\left\lvert\frac{2z-i}{z+1}\right\rvert = m$, where $i = \sqrt{-1}$, $m \in R^+$ is	(r)	an ellipse
(D)	If $\lvert\bar{z}\rvert = 25$, then the locus of the points represented by $-1 + 75\bar{z}$ is	(s)	a hyperbola

SECTION-V

Integer Answer Type

60. If imaginary part of z^2 is 16, then locus of 'z' is a conic whose latus rectum is

61. If $\lvert z_1 - z_2\rvert = \sqrt{25 - 12\sqrt{3}}$ and $\frac{z_1 - z_3}{z_2 - z_3} = \frac{3}{4} e^{\frac{i\pi}{6}}$, then area of triangle whose vertices are represented by z_1, z_2, z_3 is

62. Two lines $zi - \bar{z}i + 2 = 0$ and $z(1+i) + \bar{z}(1-i) + 2 = 0$ intersect at point 'P'. There is a complex number $\alpha = x + iy$ at a distance of 2 units from the point P which lies on the $z(1+i) + \bar{z}(1-i) + 2 = 0$. Then $[|x|]$ is ($[\cdot]$ is greatest integer function)

63. If $|z - 3i| = 3$ and $\arg z \in \left(0, \frac{\pi}{2}\right)$ then $\left|\cot(\arg z) - \frac{6}{z}\right| =$

64. If $\omega \neq 1$ is a complex cube root of unity, the least value of $n \in N$ for which $(1 + \omega^2)^n = (1 + \omega^4)^n$ is satisfied, is

65. If $iz^3 + z^2 - z + i = 0$ then $[|z|] =$ where [.] is G.I.F.

66. The maximum possible distance from $z_1 = 5 + 4i$ in the complex plane to some point on the curve $|z - 2| = 1$ is

67. The value of $e^{2mi\cot^{-1}p} \cdot \left(\frac{pi+1}{pi-1}\right)^m$ is

68. If $x = a + b\omega + c\omega^2$, $y = a\omega + b\omega^2 + c$, $z = a\omega^2 + b + c\omega$, then $\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} =$

69. If $|z_1| = 2$, $|z_2| = 3$, $|z_3| = 4$ and $|2z_1 + 3z_2 + 4z_3| = 9$, then absolute value of $8z_2z_3 + 27z_3z_1 + 64z_1z_2 - 210$ must be equal to

SOLUTIONS

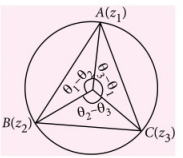
1. (b) : Let C and D divides AB in the ratio 2 : 1 internally and externally respectively.

$\therefore\ C(z_3)=\dfrac{7}{3}, D(z_4)=-3+4i$ and

$AB=|z_1-z_2|=5,\ CD=|z_3-z_4|=\dfrac{20}{3}$

$\therefore$ Max. area of $\Delta PAB=\dfrac{1}{2}\cdot\dfrac{CD}{2}\cdot AB=\dfrac{25}{3}$

2. (b) :



Since $|z_1|=|z_2|=|z_3|=1$ and $z_1+z_2+z_3=0$

$\therefore$ Triangle ABC is an equilateral triangle

$\therefore\ \theta_1-\theta_2=\theta_2-\theta_3=\theta_3-\theta_1=\dfrac{2\pi}{3}$

$\therefore\ \cos(\theta_1-\theta_2)+\cos(\theta_2-\theta_3)+\cos(\theta_3-\theta_1)=-\dfrac{3}{2}$

3. (a) : We have, $z^3=-w^5 \Rightarrow |z|^3=|w|^5$

$\Rightarrow |z|^6=|w|^{10}$ ... (1)

and $z^2=\dfrac{1}{\bar{w}^4} \Rightarrow |z|^2=\dfrac{1}{|\bar{w}|^4}$

$\Rightarrow |z|^6=\dfrac{1}{|w|^{12}} \quad (\because |w|=|\bar{w}|)$ ...(2)

From (1) and (2), we have $|w|=1$ and $|z|=1$

$\Rightarrow z\bar{z}=w\bar{w}=1$

Again $z^6=w^{10}$ ... (3)

and $z^6\cdot\bar{w}^{12}=1$

$\Rightarrow z^6=\dfrac{1}{\bar{w}^{12}}=w^{10}$ (from 3)

$\Rightarrow (w\bar{w})^{10}(\bar{w})^2=1 \Rightarrow (\bar{w})^2=1$

$\Rightarrow \bar{w}=1$ or $-1 \Rightarrow w=1$ or -1

If $w=1$, then $z^3+1=0$ and $z^2=1 \Rightarrow z=-1$

If $w=-1$, then $z^3-1=0$ and $z^2=1 \Rightarrow z=1$

Hence, $z=1$ and $w=-1$ or $z=-1$ and $w=1$

4. (b) : Let $A=x+iy$; $|A|=1 \Rightarrow x^2+y^2=1$ ...(i)

and $|A+1|=1 \Rightarrow (x+1)^2+y^2=1$ ...(ii)

From (i) and (ii), we get

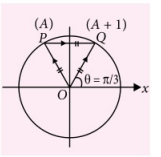
$\Rightarrow x=-\dfrac{1}{2}$ and $y=\pm\dfrac{\sqrt{3}}{2} \Rightarrow A=\omega$ or ω^2

as $A=\dfrac{-1\pm i\sqrt{3}}{2}$

$\Rightarrow (\omega)^n=(1+\omega)^n=(-\omega^2)^n$

$\Rightarrow n$ must be even and divisible by 3

Second solution : $A=\dfrac{-1+i\sqrt{3}}{2}$; $A+1=\dfrac{1+i\sqrt{3}}{2}$



and $n\arg(A)=n\arg(A+1)=\arg 1=2n\pi$

$\Rightarrow \arg A=\arg(A+1)=2\pi$

Now let $\overrightarrow{OQ}=A+1$ and $\overrightarrow{OP}=A$

$\Rightarrow \overrightarrow{OP}$ and $\overrightarrow{OQ}$ vectors must be turned a minimum number of times to coincide with positive x-axis

$\Rightarrow n=6$

5. (d) : $(z-z_1)(\bar{z}-\bar{z}_1)=(z-z_2)(\bar{z}-\bar{z}_2)$

$\Rightarrow -z\bar{z}_1-z_1\bar{z}+z_1\bar{z}_1=-z\bar{z}_2-z_2\bar{z}+z_2\bar{z}_2$

$\Rightarrow z(\bar{z}_2-\bar{z}_1)+(z_2-z_1)\bar{z}+|z_1|^2-|z_2|^2=0$

or $\bar{\alpha}z+\alpha\bar{z}+r=0$

$\Rightarrow \alpha=z_2-z_1$ and $r=|z_1|^2-|z_2|^2$

6. (a) : As α is the fifth non-real root of unity

$\therefore\ \alpha^4+\alpha^3+\alpha^2+\alpha+1=0$

Also, β is the fourth non-real root of unity

$\therefore\ \beta^3+\beta^2+\beta+1=0$

Now,

$(1+\alpha)(1+\alpha^2)(1+\alpha^4)(1+\beta)(1+\beta^2)(1+\beta^3)$

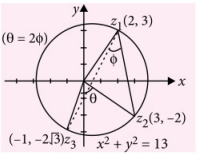
$=(1+\alpha+\alpha^2+\alpha^3)(1+\alpha^4)(1+\beta+\beta^2+\beta^3)(1+\beta^3)$

$=0$

7. (c) : Note that $|z_1|=|z_2|=|z_3|=\sqrt{13}$

Hence z_1, z_2, z_3 lies on a circle with centre $(0, 0)$ and $r=\sqrt{13}$ as shown in the figure.

Now, $\arg\left(\dfrac{z_2}{z_3}\right)=2\arg\left(\dfrac{z_2-z_1}{z_3-z_1}\right)$



$$\therefore \arg\frac{z_3}{z_2} = 2\arg\frac{z_3 - z_1}{z_2 - z_1}$$

8. (b) : Use the quadratic formula to obtain

$$z = \frac{3i \pm \sqrt{-9 + 40k}}{20}$$

which has discriminant $D = -9 + 40k$.

If $k = 1$, then $D = 31$, so (a) is false.

If k is a negative real number, then D is a negative real number, so (b) is true.

If $k = i$, then $D = -9 + 40i = 16 + 40i + 25i^2$

$$= (4 + 5i)^2$$

and the roots are $\frac{1}{5} + \frac{2}{5}i$ and $-\frac{1}{5} - \frac{1}{10}i$, so (c) is false.

If $k = 0$ (which is a complex number), then the roots are 0 and $\frac{3i}{10}$, so (d) is false.

9. (a) : $z = \sqrt{-1 - z} \Rightarrow z^2 + z + 1 = 0$

$\Rightarrow z = \omega$ or ω^2

10. (b) : $\left(2(\sqrt[3]{3})\left(\frac{1 + i\sqrt{3}}{2}\right)\right)^3 = 8 \times 3(-\omega^2)^3 = -2^4$

11. (b) : $x = \frac{x^3}{x^2} = \frac{2 + 11i}{3 + 4i} \times \frac{3 - 4i}{3 - 4i}$

$$= \frac{6 + (33 - 8)i - 44i^2}{9 - 16i^2} = \frac{(6 + 44) + 25i}{9 + 16}$$

$$= \frac{50 + 25i}{25} = 2 + i$$

So, $a + b = 2 + 1 = 3$

12. (b) : Let $z = \alpha$ be a real root

$\alpha^3 + (3 + 2i)\alpha + (-1 + ia) = 0$

$\Rightarrow (\alpha^3 + 3\alpha - 1) + i(a + 2\alpha) = 0$

$\Rightarrow \alpha^3 + 3\alpha - 1 = 0$ ... (i) and $\alpha = -a/2$ ...(ii)

Using (ii) in (i), we get

$\Rightarrow -\frac{a^3}{8} - \frac{3a}{2} - 1 = 0$ or $a^3 + 12a + 8 = 0$

Let $f(a) = a^3 + 12a + 8$

$\therefore f(-1) < 0$ and $f(0) > 0$, hence $a \in (-1, 0)$

13. (d) : If $\text{Im}(z_1 z_2) \neq 0 \Rightarrow z_1$ and z_2 are not conjugates of each other. A quadratic equation having complex roots will have real coefficients if and only if the roots are conjugates of each other. So, option (d) is incorrect.

14. (d) : $|z_1| = \left|\frac{1 - i}{1 + i\sqrt{3}}\right|^{1/6} = \left(\frac{\sqrt{2}}{2}\right)^{1/6} = 2^{-1/12}$

Similarly,

$\therefore |z_2| = 2^{-1/12}$; $|z_3| = 2^{-1/12}$; $\Rightarrow |z_1|^4 + |z_2|^4 = |z_3|^8$

15. (d) : If $b^2 - 4ac < 0$, both roots are complex numbers.

$\therefore |z|^2 = \frac{c}{a} \Rightarrow (|\alpha| = |\beta|) = \sqrt{\frac{c}{a}}$

$$\lim_{n \to \infty} f(n) = \lim_{n \to \infty} \sum_{r=1}^{n} |\alpha|^r + |\beta|^r = \frac{2 \cdot \sqrt{\frac{c}{a}}}{1 - \sqrt{\frac{c}{a}}}$$

16. (b) : $(z - a)(\bar{z} - b) = z\bar{z} - a\bar{z} - bz + ab$

$(\because z = x + iy)$

$\therefore$ Argument of

$$(z - a)(\bar{z} - b) = \tan^{-1}\left(\frac{y(a - b)}{x^2 + y^2 - ax - bx + ab}\right)$$

Also, $\frac{(\sqrt{3} + i)(1 + \sqrt{3}i)}{1 + i} = \frac{\sqrt{3} - \sqrt{3} + i(1 + 3)}{1 + i} = 2 + 2i$

$\therefore$ Argument of $2 + 2i = \tan^{-1}(2/2) = \pi/4$

and $\arg(z - a)(\bar{z} - b) = \frac{\pi}{4}$

$\Rightarrow x^2 + y^2 - (a + b)x - (a - b)y + ab = 0$

Thus, the locus of z is a circle.

Given centre is $\frac{3 + i}{2}$

$$\Rightarrow \frac{a + b}{2} = \frac{3}{2} \text{ and } \frac{a - b}{2} = \frac{1}{2} \Rightarrow a = 2, b = 1$$

17. (b) : $a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

Using the given $a + b + c = e^x$,

$a + b\omega + c\omega^2 = e^{\omega x}$, $a + b\omega^2 + c\omega = e^{\omega^2 x}$

we get $a^3 + b^3 + c^3 - 3abc = e^0 = 1$

18. (c) : If D is represented by z_4, then

$z_4 = z_2 e^{i\pi/3} + (1 - e^{i\pi/3})z_3$

$\therefore \quad 3p = (1 + e^{i\pi/3})z_2 + (2 - e^{-i\pi/3})z_3$

$3q = (1 + e^{i\pi/3})z_3 + (2 - e^{i\pi/3})z_1$

$3(p - q) = (e^{i\pi/3} - 2)z_1 + (1 + e^{i\pi/3})z_2 + (1 - 2e^{i\pi/3})z_3$

$\therefore \quad 3|p - q|^2 = \Sigma|z_1|^2 + 2\Sigma \mathrm{Re}(\omega z_1, \bar{z}_2)$

19. (c) : Let $(a + ib) = re^{i\theta}$ and $(a + ib)^{c + id} = R$ so that

$(c + id)\log(a + ib) = \log R$ or $(c + id)\log(re^{i\theta}) = \log R$

$\Rightarrow \quad (c + id)(\log r + i\theta) = \log R$

$\Rightarrow \quad c\log r - d\theta = \log R$ and $d\log r + c\theta = 0$

Hence, $\left(\frac{c^2 + d^2}{c}\right)\log r = \log R$

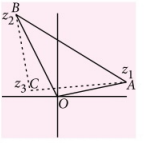
or $\left(\frac{c^2 + d^2}{c}\right)\log\left(\sqrt{a^2 + b^2}\right) = \log R$

$\Rightarrow \quad R = (a^2 + b^2)^{\frac{c^2 + d^2}{2c}} = \sqrt[2c]{(a^2 + b^2)^{(c^2 + d^2)}}$

20. (d) : $|z_2 + iz_1| = |z_1| + |z_2|$

$\Rightarrow \quad \mathrm{Arg}(iz_1) = \mathrm{Arg}(z_2)$

$\Rightarrow \quad \mathrm{Arg}(z_2) - \mathrm{Arg}(z_1) = \frac{\pi}{2}$



Let $z_3 = \frac{z_2 - iz_1}{1 - i}$

$(1 - i)z_3 = z_2 - iz_1$

$\Rightarrow \quad (z_3 - z_2) = i(z_3 - z_1) \quad \Rightarrow \quad (z_2 - z_3) = i(z_1 - z_3)$

$\therefore \quad \angle ACB = \frac{\pi}{2}$ and $|z_2 - z_3| = |z_1 - z_3|$

$\Rightarrow \quad AC = \frac{5}{\sqrt{2}}. \qquad (\because \ AB = 5)$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}AC \cdot BC = \frac{AC^2}{2} = \frac{25}{4}$ sq. units

21. (c) : $z = \frac{at + b}{t - c} \quad \Rightarrow \quad t = \frac{b + cz}{z - a}$

Now, $|t| = 1 \Rightarrow \ t\bar{t} = 1 \Rightarrow \ \frac{(b + cz)(\bar{b} + \bar{c}\,\bar{z})}{(\bar{z} - \bar{a})(z - a)} = 1$

$\Rightarrow \quad z\bar{z}(c\bar{c} - 1) + (c\bar{b} + \bar{a})z + (\bar{c}b + a)\bar{z} + b\bar{b} - a\bar{a} = 0$

22. (c) : $z^4 + 4z^3 i + 6z^2 i^2 + 4zi^3 + i^4 = 1 + i$

$\Rightarrow \quad (z + i)^4 = 1 + i \ \Rightarrow \ |z + i|^4 = \sqrt{2} \ \Rightarrow \ |z + i| = 2^{1/8}$

$|z + i| = 2^{1/8}$

Area $= \frac{d^2}{2} = \frac{4|z + i|^2}{2} = 2 \cdot 2^{1/8} \cdot 2^{1/8} = 2^{5/4}$

23. (a) : $w = \frac{1}{1 - z} = \frac{1}{(1 - \cos\theta) - i\sin\theta}$

$= \frac{1}{2\sin^2(\theta/2) - 2i\sin(\theta/2)\cos(\theta/2)}$

$= \frac{1}{2\sin(\theta/2)[\sin(\theta/2) - i\cos(\theta/2)]}$

$= \frac{\sin(\theta/2) + i\cos(\theta/2)}{2\sin\theta/2} = \frac{1}{2} + \frac{1}{2}\cot(\theta/2)i$

Hence, $\mathrm{Re}(w) = \frac{1}{2}$

$\therefore \quad w$ moved on the line $2x - 1 = 0$ parallel to y-axis.

24. (d) : Given $|z - |z + 1||^2 = |z + |z - 1||^2$

$\therefore \quad (z - |z + 1|)(\bar{z} - |z + 1|) = (z + |z - 1|)(\bar{z} + |z - 1|)$

$\Rightarrow \quad z\bar{z} - z|z + 1| - \bar{z}|z + 1| + |z + 1|^2$

$\qquad = z\bar{z} + z|z - 1| + \bar{z}|z - 1| + |z - 1|^2$

$\Rightarrow \quad |z + 1|^2 - |z - 1|^2 = (z + \bar{z})[|z - 1| + |z + 1|]$

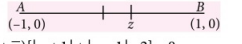
$\Rightarrow \quad (z + 1)(\bar{z} + 1) - (z - 1)(\bar{z} - 1)$

$\qquad = (z + \bar{z})[|z - 1| + |z + 1|]$

$\Rightarrow \quad (z\bar{z} + z + \bar{z} + 1) - (z\bar{z} - z - \bar{z} + 1)$

$\qquad = (z + \bar{z})[|z - 1| + |z + 1|]$

$\Rightarrow \quad 2(z + \bar{z}) = (z + \bar{z})[|z + 1| + |z - 1|]$



$(z + \bar{z})[|z + 1| + |z - 1| - 2] = 0$

$\Rightarrow \quad$ either $z + \bar{z} = 0 \ \Rightarrow \ z$ is purely imaginary

$\Rightarrow \quad z$ lies on y axis $\quad \Rightarrow \quad x = 0$

or $|z + 1| + |z - 1| = 2$

$\Rightarrow \quad z$ lies on the line segment joining $(-1, 0)$ and $(1, 0)$.

25. (a, b) : $\frac{z - 4}{z - 2i} + \frac{\bar{z} - 4}{\bar{z} + 2i} = 0$

$\Rightarrow \quad z\bar{z} + (-2 + i)z + (-2 - i)\bar{z} = 0$

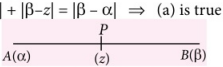
which is a circle $x^2 + y^2 - 4x - 2y = 0$

Let (x_1, y_1) be the mid-point of a chord, then its equation is

$xx_1 + yy_1 - 2(x + x_1) - (y + y_1) = x_1^2 + y_1^2 - 4x_1 - 2y_1$
It passes through (0, 0). So, the locus of (x_1, y_1) is $x^2 + y^2 - 2x - y = 0$. So, $z_1 = x_1 + iy_1$ lies on this circle for which the points (2, 0) and (0, 1) are extremities of a diameter. Also (0, 0) and (2, 1) represent extremities of another diameter.

26. (a, b, c, d) : $AP + PB = AB$

$\Rightarrow |z - \alpha| + |\beta - z| = |\beta - \alpha| \Rightarrow$ (a) is true



Now, $z = \alpha + t(\beta - \alpha)$

$= (1 - t)\alpha + t\beta$, where $t \in (0, 1) \Rightarrow$ (b) is true

Again, $\dfrac{z-\alpha}{\beta-\alpha}$ is real $\Rightarrow \dfrac{z-\alpha}{\beta-\alpha} = \dfrac{\bar{z}-\bar{\alpha}}{\bar{\beta}-\bar{\alpha}}$

$$\Rightarrow \begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0 \Rightarrow \text{(c) is true}$$

Also, $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$ if and only if $\begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} & 0 \\ \alpha & \bar{\alpha} & 1 \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} & 0 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0 \Rightarrow \text{(d) is true}$$

27. (a, c) : $z_1\bar{z}_2 = z_2\bar{z}_3 = z_3\bar{z}_1$ .... (1)

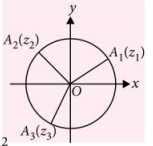
$\therefore |z_1||z_2| = |z_2||z_3| = |z_3||z_1|$

$$\therefore \frac{1}{|z_3|} = \frac{1}{|z_1|} = \frac{1}{|z_2|}$$

$\therefore |z_1| = |z_2| = |z_3| = r$

$\Rightarrow z_1z_2z_3$ lie on a circle of radius r.

$\Rightarrow |z_1|^2 = r^2 \Rightarrow \bar{z}_1 = \dfrac{r^2}{z_1}$;



Similarly,

$$\bar{z}_2 = \frac{r^2}{z_2}; \quad \bar{z}_3 = \frac{r^2}{z_3}$$

From (1), $z_1\dfrac{r^2}{z_2} = z_2\dfrac{r^2}{z_3} = z_3\dfrac{r^2}{z_1}$

$$\Rightarrow \frac{z_1}{z_2} = \frac{z_2}{z_3} = \frac{z_3}{z_1}$$

Hence, amp z_1 – amp z_2 = amp z_2 – amp z_3

= amp z_3 – amp z_1

$\Rightarrow A_1A_2A_3$ is an equilateral triangle.

$\therefore$ (2) $\Rightarrow$ (1) $\Rightarrow$ (a) is true.

and $\dfrac{z_1}{z_2} = \dfrac{z_2}{z_3} = \dfrac{z_3}{z_1}$

hence $z_2^2 = z_3z_1$ and $z_3^2 = z_1z_2$ and $z_1^2 = z_2z_3$

$\therefore$ (2) $\Rightarrow$ (3) $\Rightarrow$ (c) is true.

28. (c, d) : Here, $|z_1| = |z_2| = |z_3| = 1$

but, $|z| = 1 \Rightarrow z\bar{z} = 1$

Also given $z_1^3 + z_2^3 + z_3^3 + z_1z_2z_3 = 0$

$\Rightarrow z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 = -4z_1z_2z_3$

$\Rightarrow (z_1 + z_2 + z_3)(z_1^2 + z_2^2 + z_3^2 - \Sigma z_1z_2) = -4z_1z_2z_3$

Let $z_1 + z_2 + z_3 = z$

$$\Rightarrow \bar{z}_1 + \bar{z}_2 + \bar{z}_3 = \bar{z} = \frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}$$

$\therefore \Sigma z_1(\Sigma z_1^2 - \Sigma z_1z_2) = -4z_1z_2z_3$

$\Rightarrow \Sigma z_1((\Sigma z_1)^2 - 3\Sigma z_1z_2) = -4z_1z_2z_3$

$\Rightarrow z(z^2 - 3\Sigma z_1z_2) = -4z_1z_2z_3$

$$\Rightarrow z^3 = z_1z_2z_3\left(3z\Sigma\frac{1}{z_1} - 4\right)$$

$\Rightarrow z^3 = z_1z_2z_3(3z\bar{z} - 4) \Rightarrow |z|^3 = 3|z|^2 - 4|$

If $|z| \geq \dfrac{2}{\sqrt{3}}$, then $|z|^3 = 3|z|^2 - 4$

$\Rightarrow |z|^3 - 3|z|^2 + 4 = 0$

$\Rightarrow (|z| - 2)^2(|z| + 1) = 0$

$\Rightarrow |z| = 2 \Rightarrow |z_1 + z_2 + z_3| = 2$

If $0 \leq |z| \leq \dfrac{2}{\sqrt{3}}$, then $|z|^3 = 4 - 3|z|^2$

$\Rightarrow |z|^3 + 3|z|^2 - 4 = 0$

$\Rightarrow (|z| - 1)(|z| + 2)^2 = 0$

$\Rightarrow (|z| - 1)(|z| + 2)^2 = 0$

$\Rightarrow |z| = 1 \Rightarrow |z_1 + z_2 + z_3| = 1$

29. (a, b, c, d) : $x^{10} + (13x - 1)^{10} = 0$

$\Rightarrow x^{10} = -(13x - 1)^{10}$

$$\Rightarrow \left(\frac{13x-1}{x}\right)^{10} = -1 \Rightarrow 13 - \frac{1}{x} = (-1)^{1/10}$$

$$\Rightarrow \frac{1}{x} = 13 - (-1)^{1/10} = 13 - \text{cis}\left(\frac{2k\pi + \pi}{10}\right),$$

$k = 0, 1, 2, ..., 9$

$\therefore \quad \frac{1}{z_1} = 13 - \text{cis}\left(\frac{\pi}{10}\right), \ \frac{1}{\bar{z}_1} = 13 - \text{cis}\left(\frac{-\pi}{10}\right)$

$\Rightarrow \quad \frac{1}{|z_1|^2} = 170 - 26\cos\frac{\pi}{10}$

$\therefore \quad N = \sum_{r=1}^{5} \frac{1}{|z_1|^2}$

$= 5(170) - 26\left(\cos\frac{\pi}{10} + \cos\frac{3\pi}{10} + \cos\frac{5\pi}{10} + \cos\frac{7\pi}{10} + \cos\frac{9\pi}{10}\right)$

$= 850$ (If $\alpha + \beta = \pi$, then $\cos\alpha + \cos\beta = 0$)

30. (a, b) : $(z^p - 1)(z^q - 1) = 0$

$\Rightarrow \quad z = 1^{1/p}, \ 1^{1/q}$, but not both at the same time

31. (a, c) : $(x + (x - 2)i)(3 - i)$

$+ (2y + (1 - 3y)i)(3 + i) = 10i$

$\Rightarrow \quad 4x - 2 + 9y - 1 = 0$ and $2x - 6 + 3 - 7y = 10$

$\Rightarrow \quad 4x + 9y = 3$ and $2x - 7y = 13 \ \Rightarrow \ y = -1, x = 3$

32. (a, b) : $2\arg z^{3/8} = \arg(z^2 + \bar{z}\sqrt{z})$

$\Rightarrow \quad \frac{z^2 + \bar{z}\sqrt{z}}{z^{3/4}}$ is real

$\Rightarrow \quad z^{5/4} + \bar{z} \cdot z^{\frac{-1}{4}} = \bar{z}^{\frac{5}{4}} + z \cdot \bar{z}^{\frac{-1}{4}}$

$\Rightarrow \quad (z^{5/4} - \bar{z}^{5/4}) = \frac{z^{5/4} - \bar{z}^{(-5/4)}}{(z\bar{z})^{1/4}}$

$\Rightarrow \quad z^{5/4} = \bar{z}^{5/4}$ or $\frac{1}{(z\bar{z})^{1/4}} = 1$

and hence $z = \bar{z}$ or $z\bar{z} = 1$

33. (a, d) : Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then $(z_1 + z_2) = |x_1 + x_2| + |y_1 + y_2| \le (z_1) + (z_2)$

34. (b, c) : $z_3 = z_1(z_2 - z_1)e^{\pm i\frac{\pi}{3}}$

$= (1 + i)(-2(1 + i))\left(\frac{1 \pm \sqrt{3}i}{2}\right) = (1 + i)(1 - (1 \pm \sqrt{3}i))$

$= \pm(1 + i)(\sqrt{3}i) = \pm\sqrt{3}(i - 1)$

$z_3 = \sqrt{3}(1 - i)$ or $\sqrt{3}(i - 1)$

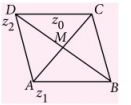
35. (a, d) : Let $A = z_1$, $z_0 = 2 - i$, $z_2 = (1 + i)$. Then

$\frac{z_1 - z_0}{z_2 - z_0} = \frac{1}{2}e^{\pm i\pi/2} = \pm\frac{i}{2}$

$\Rightarrow \quad z_1 = z_0 \pm (z_2 - z_0)\frac{i}{2}$

$= (2 - i) \pm \frac{i}{2}(-1 + 2i)$

$\Rightarrow \quad z_1 = 1 - \frac{3i}{2}$ or $3 - \frac{i}{2}$



36. (a, b, c) : $\sqrt{z_1}(1 - |z_2|) = \sqrt{z_2}i(1 + |z_1|)$

$\Rightarrow \quad \frac{\sqrt{z_1}}{\sqrt{z_2}}$ is purely imaginary

$\Rightarrow \quad \frac{z_1}{z_2}$ is purely real (negative)

$\Rightarrow \quad \frac{\sqrt{z_1}}{\sqrt{z_2}} + \frac{\sqrt{\bar{z}_1}}{\sqrt{\bar{z}_2}} = 0$

Also $\sqrt{z_1} - i\sqrt{z_2} = \sqrt{z_2\bar{z}_2}\sqrt{z_1} + i\sqrt{z_1\bar{z}_1}\sqrt{z_2}$

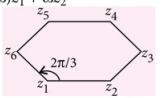
$\Rightarrow \quad \sqrt{z_1} - i\sqrt{z_2} - i\sqrt{\bar{z}_1 z_2}(\sqrt{z_1} - i\sqrt{z_2}) = 0$

$\Rightarrow \quad \left(1 - i\sqrt{\bar{z}_1 z_2}\right)\left(\sqrt{z_1} - i\sqrt{z_2}\right) = 0$

$\Rightarrow \quad \bar{z}_1 z_2 = -1$ or $\frac{z_1}{z_2} = -1$

37. (a, b, c, d) : $z_6 - z_1 = (z_2 - z_1)e^{i2\pi/3}$

$\Rightarrow \quad z_6 = (1 - \omega)z_1 + \omega z_2$



$z_3 - z_2 = (z_1 - z_2)\, e^{i4\pi/3} = (z_1 - z_2)\omega^2$

Also $z_4 = z_3 + z_6 - z_1$

38. (a, b, d) : $z_1 = 5 + 12i$, $|z_2| = 4$

(a) $|z_1 + iz_2| \le |z_1| + |iz_2| = 13 + 4 = 17$

(b) $|z_1 + (1 + i)z_2| \ge ||z_1| - |(1 + i)z_2|| \ge 13 - 4\sqrt{2}$

(c) $\left|z_2 + \frac{4}{z_2}\right| \le |z_2| + \left|\frac{4}{|z_2|}\right| = 5$

and $\left|z_2 + \frac{4}{z_2}\right| \ge \left||z_2| - \left|\frac{4}{z_2}\right|\right| = |4 - 1| = 3$

$\therefore \quad \left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right|$ has minimum 13 and maximum $\frac{13}{3}$

39. (a, b, c, d) : $az^2 + z + 1 = 0$ ..... (i)

Taking conjugate on both sides of (i), we get

$\bar{a}z^2 - z + 1 = 0$ ..... (ii) $(\because\ \bar{z} = -z)$

Eliminating 'z' from (i) and (ii), we get

$(\bar{a} - a)^2 + 2(a + \bar{a}) = 0$

$\Rightarrow \left(\frac{\bar{a} - a}{2}\right)^2 + \left(\frac{a + \bar{a}}{2}\right) = 0$

$\Rightarrow -\left(\frac{a - \bar{a}}{2i}\right)^2 + \left(\frac{a + \bar{a}}{2}\right) = 0$

$\Rightarrow -\sin^2\theta + \cos\theta = 0 \Rightarrow \cos\theta = \sin^2\theta$

Now $f(x) = x^3 - 3x^2 + 3(1 + \cos\theta)x + 5$

$\Rightarrow f'(x) = 3x^2 - 6x + 3(1 + \cos\theta)$

Its discriminant

$D = 36 - 36(1 + \cos\theta) = -36\cos\theta$

$= -36\sin^2\theta < 0$

$\Rightarrow f'(x) > 0\ \forall\ x \in R$

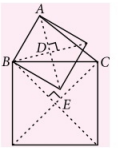
$\Rightarrow f(x)$ is increasing $\forall\ x \in R$

40. (a) : $\frac{z_B - z_D}{z_A - z_D} = \frac{BD}{AD} e^{i\pi/2} = i$

$\Rightarrow z_B - z_D = z_A i - z_D i$

$\Rightarrow z_B - z_A i = z_D(1 - i)$

$\Rightarrow z_D = \frac{z_B - z_A i}{1 - i}$



Similarly, $z_E = \frac{z_B - z_C i}{1 - i}$

$z_E - z_D = \frac{i(z_A - z_C)}{1 - i}$

Angle between AC and $DE = \arg\left(\frac{z_C - z_A}{z_E - z_D}\right)$

$= \arg\left(\frac{(z_C - z_A)(1 - i)}{i(z_A - z_C)}\right) = \arg(1 + i) = \frac{\pi}{4}$

41. (a) : $DE = |z_E - z_D| = \left|\frac{i(z_A - z_C)}{1 - i}\right|$

$= \frac{|1 - \omega^2|}{\sqrt{2}} = \sqrt{\frac{3}{2}}$

42. (a) : $z_E = \frac{z_B - z_C i}{1 - i} = \frac{\omega - \omega^2 i}{1 - i}$

$= \frac{(\omega + \omega^2) + i(\omega - \omega^2)}{2} = \frac{-1 - \sqrt{3}}{2}$

$AE = \left|1 + \frac{1 + \sqrt{3}}{2}\right| = \frac{3 + \sqrt{3}}{2}$

43. (c) $7\theta = (2n + 1)\pi,\ n = 0$ to 6

$4\theta = (2n + 1)\pi - 3\theta$

$\Rightarrow \cos 4\theta = -\cos 3\theta$

$\Rightarrow 2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta)$

$\Rightarrow 2(2x^2 - 1)^2 - 1 = -(4x^3 - 3x)$, where $x = \cos\theta$

$\Rightarrow 8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0$

$(x + 1)(8x^3 - 4x^2 - 4x + 1) = 0$

44. (a)

The roots are $\cos\frac{\pi}{7}, \cos\frac{2\pi}{7}, \ldots\ldots, \cos\frac{13\pi}{7}$, where

$\cos\frac{\pi}{7} = \cos\frac{13\pi}{7}, \cos\frac{3\pi}{7} = \cos\frac{11\pi}{7}, \cos\frac{5\pi}{7} = \cos\frac{9\pi}{7}$

$\therefore$ The roots of $8x^3 - 4x^2 - 4x + 1 = 0$ are

$\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$

45. (c)

$\sec\frac{\pi}{7}, \sec\frac{3\pi}{7}, \sec\frac{5\pi}{7}$ are roots of $\frac{8}{x^3} - \frac{4}{x^2} - \frac{4}{x} + 1 = 0$

$\Rightarrow x^3 - 4x^2 - 4x + 8 = 0$

$\therefore \sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7} = 4$

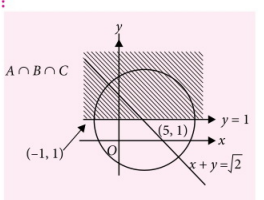
$\sec^2\frac{\pi}{7}, \sec^2\frac{3\pi}{7}, \sec^2\frac{5\pi}{7}$ are roots of $f(\sqrt{x}) = 0$

$\Rightarrow (\sqrt{x})^3 - 4(\sqrt{x})^2 - 4\sqrt{x} + 8 = 0$

$\Rightarrow x^3 - 24x^2 + 80x - 64 = 0$

$\therefore \sec^2\frac{\pi}{7} + \sec^2\frac{3\pi}{7} + \sec^2\frac{5\pi}{7} = 24$

46-48 :



46. (b) : A is the set of points on and above the line $y = 1$ ...(i) in the Argand plane. B is the set of points on the circle $(x - 2)^2 + (y - 1)^2 = 9$ ...(ii) and

$C = \text{Re}(1-i)z = \text{Re}(1-i)(x+iy)$

$\Rightarrow x + y = \sqrt{2}$ ... (iii)

Hence $A \cap B \cap C$ has only one point of intersection.

47. (c) : The points $A(-1, 1)$ and $B(5, 1)$ are the extremities of the diameter of the given circle.

Hence, $|z + 1 - i|^2 + |z - 5 - i|^2 = 36$

i.e. $(PA)^2 + (PB)^2 = 36$, which lies between 35 and 39

48. (d) : $|z| - |w| < |z - w|$ and $|z - w|$ is the distance between z and w. Here, z is fixed. Hence distance between z and w would be maximum for diametrically opposite points. Therefore,

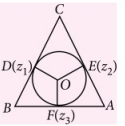
$|z - w| < 6 \Rightarrow -6 < |z| - |w| < 6$

$\Rightarrow -3 < |z| - |w| + 3 < 9$

49. (a) : Given that

$|z_1| = |z_2| = |z_3| = r$,

hence z_1, z_2, z_3 lies on a circle whose radius is r.



Clearly, $\arg\left(\frac{z_2 - z}{z_2}\right) = \frac{\pi}{2}$

$\Rightarrow \frac{z_2 - z}{z_2} + \frac{\overline{z}_2 - \overline{z}}{\overline{z}_2}$ ... (1)

and $\arg\left(\frac{z_3 - z}{z_3}\right) = -\frac{\pi}{2} \Rightarrow \quad 2r^2 - \overline{z}_3 z - z_3\overline{z} = 0$ ...(2)

50. (b) : In above solution eliminating $\overline{z}$ from equation (1) and equation (2), we get

$$(\overline{z}_2 z_3 - \overline{z}_3 z_2)z - (z_3 - z_2)2r^2 = 0$$

$$\Rightarrow \quad r^2\left(\frac{z_3}{z_2} - \frac{z_2}{z_3}\right)z - 2(z_3 - z_2)r^2 = 0$$

$$\Rightarrow \quad z = \frac{2z_2 z_3}{z_2 + z_3}$$

51. (c) : Intersesction of AO and DE is the value of z that satisfied the equation (3) and (4). Put the value of z from (3) in (4),

$$(\overline{z}_1 - \overline{z}_2)z - (z_1 - z_2)z\sqrt{\frac{\overline{z}_2\overline{z}_3}{z_2 z_3}} + (z_1\overline{z}_2 + \overline{z}_1 z_2) = 0$$

$$\Rightarrow \quad r^2\left(\frac{1}{z_1} - \frac{1}{z_2}\right)z - (z_1 - z_2)z\sqrt{\frac{\overline{z}_2\overline{z}_3}{z_2 z_3}} + r^2\left(\frac{z_1}{z_2} - \frac{z_2}{z_1}\right) = 0$$

$$\Rightarrow \quad \left(\frac{z_2 - z_1}{z_1 z_2}\right)z - \left(\frac{z_1 - z_2}{z_2 z_3}\right)z + \frac{z_1^2 + z_2^2}{z_1 z_2} = 0$$

or $z\left(\frac{1}{z_1 z_2} + \frac{1}{z_2 z_3}\right) = \frac{z_1 + z_2}{z_1 z_2}$

gives $z = \frac{(z_1 + z_2)z_3}{z_1 + z_3}$

52. (b) : $\omega_1 = \omega_2 e^{i\frac{4\pi}{3}} \Rightarrow \omega_1^3 = \omega_2^3$

$\Rightarrow \omega_1^3\overline{\omega}_1^2\overline{\omega}_2^2 = \omega_2^3\overline{\omega}_1^2\overline{\omega}_2^2 \Rightarrow \omega_2\overline{\omega}_1^2 = \omega_1\overline{\omega}_2^2$

53. (b) : Since $i\beta$ is real

$\therefore$ β is purely imaginary.

54. (c) : $-\frac{\alpha}{\overline{\alpha}} = e^{\pm i\frac{\pi}{2}} = \pm i$

$\therefore \quad (1+i)\left(-\frac{2\alpha}{\overline{\alpha}}\right) = 2(-1+i)$

55. (A) - (r), (B)- (q), (C) - (p)

(A) $\left(\frac{\omega^{n+1} - 1}{\omega - 1}\right)^m = \begin{cases} 0, & \text{if } n+1 = 3k, k \in N \text{ and } m \in N \\ 1, & \text{if } n+1 = 3k+1, k \in N \text{ and } m \in N \\ (\omega+1)^m, & \text{if } n+1 = 3k+2, k \in N \end{cases}$

Now $(\omega + 1)^m = (-\omega^2)^m$

$(-\omega^2)^m \in \{-1, \omega, \omega^2, -\omega^2\}$ depending upon the value of m.

$\Rightarrow$ Answer is 6 ; {0, 1, –1, ω, $-\omega^2$, ω^2}

(B) $\alpha_1 = 2\omega$; $\alpha_2 = 2\omega^2$

$\beta_1 = 2 + 3\omega$; $\beta_2 = 2 + 3\omega^2$

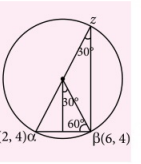
$\gamma_1 = 2 + 3\omega^2$; $\gamma_2 = 2 + 3\omega$

$\delta_1 = 3$

$\Rightarrow$ roots are 2ω, $2\omega^2$, $2 + 3\omega$, $2 + 3\omega^2$, 3

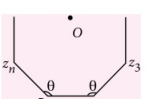
(C) Refer to the figure

$\cos 60^\circ = \frac{5}{R} \Rightarrow R = 4$



56. (A) - (r); (B) - (r) ; (C) - (q) ; (D) - (q)

Let θ be an interior angle of the polygon.



$z_n - z_1 = (z_2 - z_1)e^{i\theta}$, $z_3 - z_2 = (z_1 - z_2)e^{-i\theta}$

$\therefore \quad z_3 + z_n = z_2 + (z_1 - z_2)e^{-i\theta} + z_1 + (z_2 - z_1)e^{i\theta}$

$= (1 - 2i\sin\theta)z_1 + (1 + 2i\sin\theta)z_2$

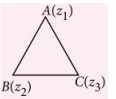
$\therefore A = 1 - 2i\sin\theta \Rightarrow |A| = \sqrt{3 - 2\cos 2\theta} = \sqrt{3 - 2\cos\frac{4\pi}{n}}$

$\therefore \quad |A| = \sqrt{5}, 2, \sqrt{3}, \sqrt{2}$ according to $n = 4, 6, 8, 12$

57. (A) - (s), (B) - (r), (C) - (q), (D) - (p)

(A) $\overline{BA} = \overline{BC}\, e^{\frac{i\pi}{3}}$

$\Rightarrow z_1 - z_2 = (z_3 - z_2)e^{i\pi/3}$ ... (1)



Again, $\overline{CA} = \overline{CB}\, e^{\frac{-i\pi}{3}}$

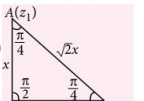
$\Rightarrow (z_1 - z_3) = (z_2 - z_3)\, e^{-i\pi/3}$ ...(2)

From (1) & (2), we have

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{z_2 - z_3}{z_1 - z_3} \Rightarrow \Sigma z_1^2 = \Sigma z_1 z_2$$

(B) $\overline{CA} = \overline{CB}\, e^{\frac{-i\pi}{2}}$

$\Rightarrow z_1 - z_3 = (z_2 - z_3)(-i)$ ... (1)



Again $\overline{BA} = \sqrt{2}\,\overline{BC}\, e^{\frac{i\pi}{4}}$

$(z_1 - z_2)^2 = 2(z_3 - z_2)^2$ (i)

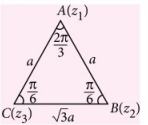
$$\Rightarrow (z_1 - z_2)^2 = 2(z_3 - z_2)^2 \frac{(-z_3 + z_1)}{(-z_2 + z_3)}$$

$= 2(z_3 - z_2)(z_1 - z_3)$

(C) $\overline{BA} = \frac{1}{\sqrt{3}}\,\overline{BC}\, e^{\frac{i\pi}{6}}$,

$\overline{CA} = \frac{-1}{\sqrt{3}}\,\overline{BC}\, e^{\frac{-i\pi}{6}}$

$\Rightarrow \overline{BC}^2 = -3\overline{BA}\cdot\overline{CA}$



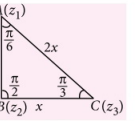
$\Rightarrow (z_2 - z_3)^2 = -3(z_3 - z_1)(z_2 - z_1)$

$= 3(z_1 - z_2)(z_3 - z_1)$

(D) $\overline{BA} = \sqrt{3}\,\overline{BC}\, e^{\frac{i\pi}{2}}$

squaring, we get

$(z_1 - z_2)^2 = 3(z_3 - z_2)^2$ (–1)



58. (A) - (s), (B) - (r), (C) - (p), (D) - (q)

(A) $|z - 1|^2 + |z + 1|^2 = 4 \Rightarrow x^2 + y^2 = 1$

$\Rightarrow r = 1$

(B) Extremities of diameter are (0, 2) and (0, –2)

$\Rightarrow$ radius = 2

(C) $z\bar{z} - z(1 - 2i) - \bar{z}(1 + 2i) - 4 = 0$

$\Rightarrow x^2 + y^2 - 2x - 4y - 4 = 0$

$\therefore$ Centre (1, 2), $r = 3$

(D) $|z + 3| = 2|z - 3| \Rightarrow x^2 + y^2 - 10x + 9 = 0$

$\Rightarrow r = 4$

59. (A) - (s) ; (B) - (r) ; (C) – (p, q) ; (D) - (q)

(A) Let $z = x + iy \Rightarrow x^2 - y^2 = 2x$

$\Rightarrow z$ lies on a hyperbola.

(B) $|z - z_1| + |z - z_2| = \lambda$ and $\lambda > |z_1 - z_2| \Rightarrow z$ lies on an ellipse.

(C) $\frac{|z - i/2|}{|z + 1|} = \frac{m}{2}$, if $m = 2$

$\Rightarrow z$ lies on a straight line

If $m \neq 2 \Rightarrow z$ lies on a circle

(D) $|\bar{z}| = 25$, let $-1 + 75\bar{z} = \omega$

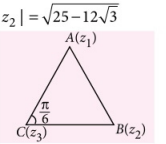
$\Rightarrow |\omega + 1| = 75|\bar{z}| = 75 \times 25$

$\Rightarrow \omega$ lies on a circle.

60. (8) : $z = x + iy \Rightarrow 2xy = 16 \Rightarrow xy = 8 = c^2$

$\therefore$ Latus rectum $= 2\sqrt{2}c = 2\sqrt{2}\cdot 2\sqrt{2} = 8$

61. (2) : $|z_1 - z_2| = \sqrt{25 - 12\sqrt{3}}$



$|z_2 - z_3| = x$, $|z_1 - z_3| = \frac{3x}{4}$

$$\Rightarrow |z_1 - z_2|^2 = x^2 + \frac{9x^2}{16} - \frac{3x^2}{2}\cos\frac{\pi}{6} = 25 - 12\sqrt{3}$$

$\Rightarrow \quad x = 4$

$\therefore \quad$ Area $= \frac{1}{2} \cdot 4 \cdot 3 \cdot \frac{1}{2} = 3$ sq. units

62. (1) : $\alpha = x + iy$

$(x + iy)i - (x - iy)i + 2 = 0 \Rightarrow y = 1$

and $z(1+i) + \bar{z}(1-i) + 2 = 0 \Rightarrow x - y + 1 = 0$

$x^2 + (y-1)^2 = 4 \Rightarrow x = \pm\sqrt{2}$ $[|x| = 1]$

63. (1) : $|z - 3i| = 3 \Rightarrow |re^{i\theta} - 3i| = 3$

$|r\cos\theta + i(r\sin\theta - 3)| = 3$

$\Rightarrow r^2\cos^2\theta + (r\sin\theta - 3)^2 = 9 \Rightarrow \sin\theta = \frac{r}{6}$

$\therefore \quad \left|\cot\theta - \frac{6}{re^{i\theta}}\right| = 1$

64. (3) : $n = 3 \Rightarrow (1+\omega^2)^3 = (1+\omega^4)^3 = -1$

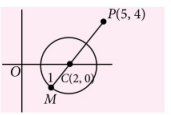
65. (1) : $iz^3 + z^2 + i^2 z + i = 0 \Rightarrow (iz+1)(z^2+i) = 0$

$\Rightarrow |z| = \left|\frac{-1}{i}\right|$ or $|\sqrt{i}| = 1$

$\therefore \quad |z| = 1$ in any case

$\therefore \quad [|z|] = 1$

66. (6) :



$PM = PC + CM = 5 + 1 = 6$

67. (1) : Let $\cot^{-1} p = \theta$, then $\cot\theta = p$

Now,

$$e^{2mi\theta} \cdot \left(\frac{i\cot\theta + 1}{i\cot\theta - 1}\right)^m = e^{2mi\theta}\left[\frac{i(\cot\theta - i)}{i(\cot\theta + i)}\right]^m$$

$$= e^{2mi\theta}\left[\frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}\right]^m$$

$$= e^{2mi\theta}\left(\frac{e^{-i\theta}}{e^{i\theta}}\right)^m = e^{2mi\theta}(e^{-2i\theta})^m$$

$$= e^{2mi\theta}e^{-2mi\theta} = e^0 = 1$$

68. (3) : From the given equation

$x + y + z = (a + b\omega + c\omega^2) + (a\omega + b\omega^2 + c) + (a\omega^2 + b + c\omega)$

$= a(1 + \omega + \omega^2) + b(\omega + \omega^2 + 1) + c(\omega^2 + 1 + \omega)$

$= a \times 0 + b \times 0 + c \times 0 = 0$

i.e. $x + y + z = 0$

$\Rightarrow \quad x^3 + y^3 + z^3 = 3xyz \Rightarrow \frac{x^3 + y^3 + z^3}{xyz} = 3$

$\Rightarrow \quad \frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = 3$

69. (6) : The absolute value of

$8z_2z_3 + 27z_3z_1 + 64z_1z_2 = |\,8z_2z_3 + 27z_3z_1 + 64z_1z_2\,|$

$= |z_1z_2z_3|\left|\frac{8}{z_1} + \frac{27}{z_2} + \frac{64}{z_3}\right|$

$= |z_1||z_2||z_3|\left|\frac{8\bar{z}_1}{z_1\bar{z}_1} + \frac{27\bar{z}_2}{z_2\bar{z}_2} + \frac{64\bar{z}_3}{z_3\bar{z}_3}\right|$

$= |z_1||z_2||z_3|\left|\frac{8\bar{z}_1}{|z_1|^2} + \frac{27\bar{z}_2}{|z_2|^2} + \frac{64\bar{z}_3}{|z_3|^2}\right|$

$= |z_1||z_2||z_3|\left|\frac{8\bar{z}_1}{4} + \frac{27\bar{z}_2}{9} + \frac{64\bar{z}_3}{16}\right|$

$= |z_1||z_2||z_3||2\bar{z}_1 + 3\bar{z}_2 + 4\bar{z}_3|$

$= |z_1||z_2||z_3|\overline{|2z_1 + 3z_2 + 4z_3|}$

$= |z_1||z_2||z_3||2z_1 + 3z_2 + 4z_3| = 2 \cdot 3 \cdot 4 \cdot 9 = 216$

$\therefore \quad 8z_2z_3 + 27z_3z_1 + 64z_1z_2 - 210 = 216 - 210 = 6$

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concept BOOSTERS

Class XII

Limit, Continuity & Differentiability

* ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

Limit of a function $f(x)$ is said to exist as, $x \to a$ when

$\underset{x\to a^-}{\text{Limit}}\, f(x) = \underset{x\to a^+}{\text{Limit}}\, f(x) =$ finite quantity.

FUNDAMENTAL THEOREMS ON LIMITS

Let $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$.

If l & m exists then :

- $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$; where k is a constant.
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{l}{m}$, provided $m \neq 0$
- $\lim_{x\to a} f(x) \pm g(x) = l \pm m$
- $\lim_{x\to a} f(x) \cdot g(x) = l \cdot m$
- $\lim_{x\to a} f[g(x)] = f\left(\lim_{x\to a} g(x)\right) = f(m)$; provided f is continuous at $g(x) = m$.

For example :

$\lim_{x\to a} \ln f(x) = \ln\left[\lim_{x\to a} f(x)\right] \ln l (l > 0)$

STANDARD LIMITS

- If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} \phi(x) = \infty$, then;

  $\lim_{x\to a} [f(x)]^{\phi(x)} = e^{\lim\limits_{x\to a} \phi(x)[f(x)-1]}$
- If $\lim_{x\to a} f(x) = A > 0$ & $\lim_{x\to a} \phi(x) = B$ (a finite quantity) then;

  $\lim_{x\to a} [f(x)]^{\phi(x)} = e^z$

  where $z = \lim_{x\to a} \phi(x).\ln[f(x)] = e^{B \ln A} = A^B$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a (a > 0)$.

  In particular $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\tan^{-1} x}{x}$

  $= \lim_{x\to 0} \frac{\sin^{-1} x}{x}$

  [where x is measured in radians]
- $\lim_{x\to 0} (1+x)^{1/x} = e = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$

  Note however there $\lim_{\substack{h\to 0 \\ n\to\infty}} (1-h)^n = 0$

  and $\lim_{\substack{h\to 0 \\ n\to\infty}} (1+h)^n \to \infty$
- $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

SQUEEZE PLAY THEOREM/ SANDWICH THEOREM

If $f(x) \le g(x) \le h(x)\ \forall\ x$ and

$\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

INDETERMINATE FORMS

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, 0°, ∞°, $\infty - \infty$ and 1^∞

OBSERVATIONS

- We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number. It does not obey the laws of elementary algebra.
- $\infty + \infty = \infty$
- $\infty \times \infty = \infty$
- $(a/\infty) = 0$, if a is finite
- $a/0$ is not defined , if $a \neq 0$.
- $ab = 0$, if & only if $a = 0$ or $b = 0$ and a & b are finite.
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\text{for} -1 < x \leq 1$
- $a^x = 1 + \frac{x \ln a}{1!} + \frac{x^2 \ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + ... \text{ for } a > 0$
- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$
- $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$
- $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$
- $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2 \cdot 3^2}{5!}x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!}x^7 + ...$
- $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + ...$

SECTION-I

Single Correct Answer Type

1. $\lim\limits_{\theta \to \frac{\pi}{2}} \left[\frac{\theta - \frac{\pi}{2}}{\cot\theta} \right]$, where [.] denotes the greatest integer function is

(a) -2 (b) 0 (c) -1 (d) 1

2. $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n^2 + \cos n}\right)^{n^2+n} =$

(a) 1 (b) e (c) $\frac{1}{e}$ (d) e^2

3. $\lim\limits_{x\to\infty} \frac{x^3 \sin\left(\frac{1}{x}\right) - 2x^2}{1+3x^2} =$

(a) 0 (b) $-\frac{1}{3}$ (c) -1 (d) $-\frac{2}{3}$

4. $\lim\limits_{x\to 0} \frac{e^x - e^{\sin x}}{2(x - \sin x)} =$

(a) $-\frac{1}{2}$ (b) 1 (c) $\frac{3}{2}$ (d) $\frac{1}{2}$

5. $\lim\limits_{h\to 0} \left[\frac{1}{h\sqrt[3]{8+h}} - \frac{1}{2h} \right] =$

(a) $-\frac{1}{48}$ (b) $-\frac{4}{3}$ (c) $-\frac{16}{3}$ (d) $\frac{1}{12}$

6. $\lim\limits_{x\to\infty} \frac{\log x^n - [x]}{[x]} =$

(a) n (b) -1 (c) 1 (d) 0

7. $\lim\limits_{x\to 2^+} \{x\} \frac{\sin(x-2)}{(x-2)^2} =$

(a) 0 (b) ∞
(c) 1 (d) does not exist

8. $\lim\limits_{x\to 0} \frac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x} =$

(a) 1 (b) 2 (c) $\log_e a$ (d) $\log_a e$

9. If $f(x) = \frac{1}{3}\left(f(x+1) + \frac{5}{f(x+2)} \right)$ and $f(x) > 0$, $\forall\, x \in R$ then $\lim\limits_{x\to\infty} f(x)$ is

(a) 0 (b) $\sqrt{\frac{2}{5}}$ (c) $\sqrt{\frac{5}{2}}$ (d) ∞

10. Let f be a differentiable function on the open interval (a, b). Which of the following statements must be true?

I. f is continuous on the closed interval $[a, b]$
II. f is bounded on the open interval (a, b)

III. If $a < a_1 < b_1 < b$, and $f(a_1) < 0 < f(b_1)$, then there is a number c such that $a_1 < c < b_1$ and $f(c) = 0$

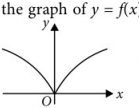
(a) I and II only (b) I and III only
(c) II and III only (d) only III

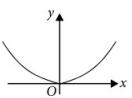
11. For the function $f(x) = \lim_{n\to\infty} \frac{1}{1+n\sin^2(\pi x)}$, which of the following holds?

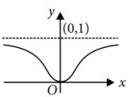
(a) The range of f is a singleton set
(b) f is continuous on R
(c) f is discontinuous for all $x \in I$
(d) f is discontinuous for some $x \in R$

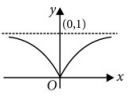
12. Let $y = f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

Then which of the following can best represent the graph of $y = f(x)$?

(a) 

(b) 

(c) 

(d) 

13. If $f''(x) > 0 \ \forall \ x \in R$ and $f'(3) = 0$ and $g(x) = f(\tan^2 x - 2\tan x + 4)$, $0 < x < \pi/2$ then $g(x)$ is increasing in

(a) $(0, \pi/3)$ (b) $(0, \pi/6)$
(c) $(0, \pi/4)$ (d) $(\pi/4, \pi/2)$

14. Let $f(x)$ be a function which is differentiable any number of times $f(2x^2 - 1) = 2x^3 f(x)$, $\forall \ x \in R$ then $f^{(2010)}(0) =$
(Here $f^{(n)}(x) = n^{\text{th}}$ order derivative of 'f' at x)

(a) -1 (b) 1
(c) 0 (d) data insufficient

15. A function $g(x) = \frac{1}{4}f(2x^2-1) + \frac{1}{2}f(1-x^2)$ and $f'(x)$ is increasing function then $g(x)$ is increasing in the interval

(a) $(-\sqrt{2/3}, \sqrt{2/3})$
(b) $(-\sqrt{2/3}, 0) \cup (\sqrt{2/3}, \infty)$
(c) $(-1, 1)$
(d) $(-\infty, -\sqrt{2/3}) \cup (\sqrt{2/3}, \infty)$

16. If $f(x) = ||\sin(|x| - 1)| - 2|$ then

(a) $f(x)$ is continuous but not differentiable at $x = 2$
(b) $f(x)$ is discontinuous at $x = 2$
(c) $f(x)$ is differentiable at $x = 2$ and $f'(2) = \cos 1$
(d) $f(x)$ is differentiable at $x = 2$ and $f'(2) = -\cos 1$

17. Let $f(x) = [n + p\sin x]$, $x \in (0, \pi)$, $n \in Z$, 'p' a prime number. The number of points at which $f(x)$ is non-differentiable, is ($[\cdot]$ represent G.I.F)

(a) p (b) $p - 1$ (c) $2p - 1$ (d) $2p + 1$

18. If $f(x)$, $g(x)$ are twice differentiable functions on $[0, 2]$ satisfying $f''(x) = g''(x)$, $f'(1) = 2$, $g'(1) = 4$ and $f(2) = 3$, $g(2) = 9$, then $f(x) - g(x)$ at $x = 4$ equals

(a) 6 (b) -10 (c) 8 (d) 2

19. Let $f''(x)$ be continuous at $x = 0$ and $f''(0) = 4$ the value of $\lim_{x\to 0} \frac{2f(x) - 3f(2x) + f(4x)}{\sin(x^2)}$ is

(a) 6 (b) 10 (c) 11 (d) 12

20. If $f(x) = (1 + x^2)\,\text{sgn}(x)$ (where $\text{sgn}(x)$ is signum function) then $f^{-1}(x)$ is

(a) discontinuous as it is inverse of a discontinuous function
(b) discontinuous only at $x = 0$
(c) continuous on its domain
(d) discontinuous at some points of its domain

21. Given that $f(x) = \begin{cases} \frac{xg(x)}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ and $g(0) = 0 = g'(0)$, then the value of $f'(0)$ is

(a) 1 (b) 0
(c) -1 (d) does not exist

SECTION-II

More than One Correct Answer Type

22. $\lim_{x\to\infty} \sqrt{x}(\sqrt{x+1} - \sqrt{x})$ is equal to

(a) $\lim_{x\to 0} \frac{\log(1+x) - x}{x^2}$ (b) $\lim_{x\to 0} \frac{1 - \cos x}{x^2}$

(c) $\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x}$ (d) $\lim_{x\to 0} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x^2 + 2x}}}$

23. If $f(x)=x\left(\dfrac{e^{|x|+[x]}-2}{|x|+[x]}\right)$, then

(a) $\lim\limits_{x\to 0^+} f(x)=-1$ (b) $\lim\limits_{x\to 0^-} f(x)=0$

(c) $\lim\limits_{x\to 0} f(x)=-1$ (d) $\lim\limits_{x\to 0} f(x)=0$

24. The function $f(x)$ is defined by $f(x)=\dfrac{[x^2]-1}{x^2-1}$

for $x^2\neq 1$ and $f(1)=f(-1)=0$, then

(a) $\lim\limits_{x\to 1^+} f(x)=0$

(b) $\lim\limits_{x\to 1^-} f(x)=1$ (c) $\lim\limits_{x\to 1^-} f(x)=0$

(d) $\lim\limits_{x\to 1} f(x)$ does not exist

25. If $\lim\limits_{x\to 0} g(x)$ exists and $\lim\limits_{x\to 0} g(x).\left(\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right)$

also exist, then $g(x)$ =

(a) x (b) $x+4$

(c) x^2 (d) $4x^2+3x+2$

26. If $\lim\limits_{x\to 0}\dfrac{ae^x+b\cos x+ce^{-x}}{e^{2x}-2e^x+1}=4$, then

(a) $a=2$ (b) $b=-4$

(c) $c=2$ (d) $a+b+c=-8$

27. α, β are the roots of $ax^2+bx+c=0$. Also,

$$L_1=\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2},$$

$$L_2=\lim_{x\to\beta}\frac{1-\cos(ax^2+bx+c)}{(x-\beta)^2} \text{ and}$$

$$L_3=\lim_{(x-\alpha)(x-\beta)\to 0}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2(x-\beta)^2}.$$

Then

(a) $L_1=L_2$ (b) $L_1=L_3(\alpha-\beta)^2$

(c) $L_2=L_3(\alpha-\beta)^2$ (d) $L_3=\dfrac{a^2}{2}$

28. If $\lim\limits_{x\to\infty} 4\times\left(\dfrac{\pi}{4}-\tan^{-1}\dfrac{x+1}{x+2}\right)=y^2+4y+5$

then y can be equal to

(a) 1 (b) -1 (c) -4 (d) -3

29. Given a real valued function f such that

$$f(x)=\begin{cases}\dfrac{\tan^2 x}{(x^2-[x]^2)}, & \text{for } x>0\\ 1, & \text{for } x=0\\ \sqrt{\{x\}}\cot\{x\}, & \text{for } x<0\end{cases}$$

Where $[x]$ is the integral part and $\{x\}$ is the fractional part of x, then

(a) $\lim\limits_{x\to 0} f(x)=1$

(b) $\lim\limits_{x\to 0^-} f(x)=\sqrt{\cot 1}$

(c) $\cot^{-1}\left(\lim\limits_{x\to 0^-} f(x)\right)^2=1$

(d) f is continuous at $x=0$

30. A function is defined as

$f(x)=[\tan x]+\sqrt{\tan x-[\tan x]}, 0\le x<\dfrac{\pi}{2}$,

where [.] denotes the greatest integer function then

(a) $f(x)$ is continuous in $\left[0,\dfrac{\pi}{2}\right)$

(b) $f(x)$ is not continuous at $x=0$

(c) $f(x)$ is continuous at $x=0$, $\dfrac{\pi}{4}$

(d) $f(x)$ has infinite points of discontinuity

31. If $f(x)=\sec 2x+\text{cosec}\,2x$, then $f(x)$ is discontinuous at all points in

(a) $\{n\pi,\ n\in N\}$ (b) $\left\{(2n\pm 1)\dfrac{\pi}{4}, n\in I\right\}$

(c) $\left\{\dfrac{n\pi}{4}, n\in I\right\}$ (d) $\left\{(2n\pm 1)\dfrac{\pi}{8}, n\in I\right\}$

32. Which is discontinuous at $x=1$?

(a) $f(x)=\dfrac{1}{1+2^{\tan x}}$

(b) $g(x)=\lim\limits_{n\to\infty}\dfrac{1}{1+n\sin^2(\pi x)}$

(c) $h(x)=2^{-2^{\left(\frac{1}{1-x}\right)}}, x\neq 1$ and $h(1)=1$

(d) $\phi(x)=\dfrac{x-1}{|x-1|+2(x-1)^2}, x\neq 1$ and $\phi(1)=1$

33. If $f(x)=\lim_{m\to\infty}\left\{\lim_{n\to\infty}\cos^{2m}(n!\pi x)\right\}$ and $g(x)=\begin{cases}0, & \text{if } x \text{ is rational}\\ 1, & \text{if } x \text{ is irrational}\end{cases}$, then

(a) f is discontinuous everywhere
(b) g is discontinuous everywhere
(c) $f+g$ is continuous everywhere
(d) Range of $f+g$ is a singleton set

34. Which is not differentiable at $x=0$?
(a) $\cos|x|+|x|$ (b) $\cos|x|-|x|$
(c) $\sin|x|+|x|$ (d) $\sin|x|-|x|$

35. The function $f(x)=\begin{cases}|x-3|, & x\geq 1\\ \left(\frac{x^2}{4}\right)-\left(\frac{3x}{2}\right)+\left(\frac{13}{4}\right), & x<1\end{cases}$ is
(a) continuous at $x=1$
(b) differentiable at $x=1$
(c) continuous at $x=3$
(d) differentiable at $x=3$

36. The function, $f(x)=[|x|]-|[x]|$, where $[x]$ denotes greatest integer function
(a) is continuous for all positive integers
(b) is discontinuous for all non positive integers
(c) has finite number of elements in its range
(d) is such that its graph does not lie above the x-axis.

SECTION-III

Comprehension Type

Paragraph for Question No. 37 to 39

If f, g, h are functions having a common domain D such that $f(x)\leq g(x)\leq h(x)$ and $\lim_{x\to a}f(x)=\lim_{x\to a}h(x)=l$ then according to Sandwich theorem $\lim_{x\to a}g(x)=l$.

Using it, compute the following limits.

37. $\lim_{x\to 0}\frac{|x|}{\sqrt{x^4+4x^2+7}}=$
(a) 1 (b) 0
(c) 1/2 (d) does not exist

38. $\lim_{x\to 0}x^2\left(\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right)=$
(a) 0 (b) ∞
(c) 1 (d) does not exist

39. $\lim_{x\to 0}x^5\left[\frac{1}{x^3}\right]$, where [.] represents greatest integer function and $f(0)=0$ is
(a) -1 (b) ∞
(c) 0 (d) does not exist

Paragraph for Question No. 40 to 42

Let $f(x)$ be a function and a be a point within the domain of $f(x)$. Then, we say that $f(x)$ has oscillating discontinuity at $x=a$ if $\lim_{x\to a^-}f(x)$ and $\lim_{x\to a^+}f(x)$ lie in a certain range but do not approach to a definite value at $x=a$. $f(x)$ has jump discontinuity at $x=a$ if $\lim_{x\to a^-}f(x)$ and $\lim_{x\to a^+}f(x)$ both exist finitely but their values are unequal.

40. The function $f(x)=\begin{cases}\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ at $x=0$
(a) is continuous
(b) has removable discontinuity
(c) has jump discontinuity
(d) has oscillating discontinuity

41. The function defined by
$f(x)=\begin{cases}\frac{x-1}{\frac{1}{e^{x-1}}+1}, & x\neq 1\\ 0, & x=1\end{cases}$ at $x=1$
(a) is continuous
(b) has removable discontinuity
(c) has jump discontinuity
(d) has finite discontinuity

42. Which of the following is true about $f(x)=\lim_{n\to\infty}\frac{\log(2+x)-x^{2n}\sin x}{1+x^{2n}}$?
(a) $\lim_{x\to 1^-}f(x)=\log 3$
(b) $\lim_{x\to 1^+}f(x)=0$
(c) $f(1)=\log 3-\sin 1$
(d) $f(x)$ has removable discontinuity at $x=1$

Paragraph for Question No. 43 to 45

It can be shown that if $f(x)$ is continuous at $x=0$ then $f(x)$ is differentiable at $x=0$. By changing

origin, we can say that if $f(x)$ is continuous at a then $(x-a)f(x-a)$ is differentiable at a.

43. The largest set over which $\frac{x\sin|x|}{1-|x|^2}$ is differentiable, is

(a) $R - \{0, 1, -1\}$ (b) R

(c) $R - \{1, -1\}$ (d) None of these

44. The number of points where the function, $(x-3)|x^2-7x+12| + \cos|x-3|$ is not differentiable is

(a) one (b) two (c) three (d) infinite

45. Let $f(x) = |x|$, $g(x) = \sin x$ and $h(x) = g(x)\,f(g(x))$ then

(a) $h(x)$ is continuous but not differentiable at $x = 0$

(b) $h(x)$ is continuous and differentiable every where

(c) $h(x)$ is continuous everywhere and differentiable only at $x = 0$

(d) None of these

SECTION-IV

Matrix-Match Type

46. Match the following.

	Column-I		Column-II
(A)	The least positive integral solution of $x^2 - 4x > \cot^{-1}x$	(p)	1
(B)	The least positive integral value of x for which $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$ is increasing	(q)	2
(C)	Let $f: R \to R$ be such that $f(a) = 1$, $f'(a) = 2$, then $\lim_{x\to 0}\left(\frac{f^2(a+x)}{f(a)}\right)^{\frac{1}{x}} = e^k$, then $k =$	(r)	5
(D)	Number of integral values of x which satisfy equation $\sin^{-1}((3-x)(x-1)) + \sin^{-1}(2-|x|) = \pi/2$ is/are	(s)	4

47.

	Column-I		Column-II
(A)	$f(x) = \begin{cases} \frac{1}{\lvert x\rvert} & \text{for } \lvert x\rvert \geq 1 \\ ax^2 + b & \text{for } \lvert x\rvert < 1 \end{cases}$ is differentiable every where and $\lvert k\rvert = a + b$, then the value of k is	(p)	2
(B)	If $f(x) = \operatorname{sgn}(x^2 - ax + 1)$ has exactly one point of discontinuity, then the value of a can be	(q)	-2
(C)	$f(x) = [2 + 3\lvert n\rvert\sin x]$, $n \in N$, $x \in (0, \pi)$ has exactly 11 points of discontinuity, then the value of n is	(r)	1
(D)	$f(x) = \lvert\lvert x\rvert - 2\rvert + a\rvert$ has exactly three points of non-differentiability, then the value of a is	(s)	-1

48. If $[x]$ denotes G.I.F. of x, and $\{x\}$ denotes fractional part of x then match the following.

	Column-I		Column-II
(A)	$(x+1)\lvert x-1\rvert$	(p)	continuous in $(-1, 1)$
(B)	$\min\{\lvert x\rvert, 1 - \lvert x\rvert\}$	(q)	differentiable in $(-1, 1)$
(C)	$\{x\} + 2[x]$, $\{.\}$ denotes fractional part of x	(r)	strictly increasing in $(-1, 1)$
(D)	$\sqrt{\cos^2\frac{\pi x}{2}}$	(s)	not differentiable at least at one point in $(-1, 1)$

SECTION-V

Integer Answer Type

49. If $\lim_{x\to 0}\frac{\cos 4x + a\cos 2x + b}{x^4}$ is finite then $b - a =$

50. Let $f(x)$ be a twice differentiable function and $f''(0) = 2$ then $\lim_{x\to 0}\frac{2f(x) - 3f(2x) + f(4x)}{x^2} =$

51. If $\lim_{n\to\infty}\{(1+x)(1+x^2)(1+x^4).....n \text{ factors}\}$ $= f(x)$ for $|x|<1$ then $f\left(\frac{1}{2}\right)=$

52. If $f(x+y)=f(x)\,f(y)$ and $f(x)=1+xg(x)\,H(x)$ where $\lim_{x\to 0} g(x)=2$, $\lim_{x\to 0} H(x)=3$ then $f'(x)=K\,f(x)$, if $K=$

53. $\lim_{x\to 0}\frac{e^{\tan x}-e^{\sin x}}{(e^{x/32}-1)(1-\cos 4x)}$

54. $\lim_{x\to 0}\frac{1-\cos^n(1-\cos x)}{\tan^m x}=1$. Then $\frac{n}{m}=$

55. If $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$ and if $f'(0)=3$, then $f\left(\tan\frac{1}{3}\right)$ must be equal to

56. If $y=e^{\sqrt{x}}+e^{-\sqrt{x}}$ and $kx\frac{d^2y}{dx^2}+2\frac{dy}{dx}-y=0$ then the value of constant k must be

57. If the function $y=f(x)$ is given by $x=3t^2$, $y=3t-t^3$ satisfies $k\frac{d^2y}{dx^2}(y-\sqrt{3x})=x+3$. Then the numerical quantity $\sqrt{k}$ should be

58. If the third order derivative of $\frac{x^4}{(x-1)(x-2)}$ is $\frac{-12k}{(x-2)^4}+\frac{6}{(x-1)^4}$ then the numerical quantity k must be equal to

59. If $y=\sqrt{2x-x^2}$ and k is a numerical quantity such that $y^3=\frac{d^2y}{dx^2}+k=0$, , then k should be

60. The function given by

$$f(x)=\begin{cases}\dfrac{\sqrt{\pi}-\sqrt{\cos^{-1}x}}{\sqrt{x+1}}, & x\neq -1\\[2ex] \dfrac{1}{\sqrt{\lambda\pi}}, & x=-1\end{cases}$$

the value of λ for which the function $f(x)$ is continuous at $x=-1$ from the right, must be

61. Let $g:(-\infty,\infty)\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ defined as $g(x)=2\tan^{-1}(e^x)-\frac{\pi}{2}$ and f is inverse of g then the value of $f'(0)$ is

SOLUTIONS

1. (c) : Let $t=\frac{\pi}{2}-\theta$ then by substitution the problem is $\lim_{t\to 0}\left[\frac{-t}{\tan t}\right]$

$\because\ 0<\frac{t}{\tan t}<1 \Rightarrow -1<\frac{-t}{\tan t}<0\ \therefore\ \left[\frac{-t}{\tan t}\right]=-1$

2. (b) : Given limit $=e^{\lim_{n\to\infty}\left(\frac{n^2+n}{n^2+\cos n}\right)}=e$

3. (b) : Required value

$$=\lim_{x\to\infty}\frac{x\sin\left(\frac{1}{x}\right)-2}{\frac{1}{x^2}+3}=\frac{1-2}{0+3}=\frac{-1}{3}$$

4. (d) : Required value

$$=\lim_{x\to 0}\frac{e^{\sin x}(e^{x-\sin x}-1)}{2\ (x-\sin x)}=\frac{1}{2}(1)=\frac{1}{2}$$

5. (a) : Required value

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sqrt[3]{8+h}}-\frac{1}{\sqrt[3]{8}}\right]$$

$$=\lim_{h\to 0}\frac{\sqrt[3]{8}-\sqrt[3]{8+h}}{h}\times\frac{1}{\sqrt[3]{8+h}\,\sqrt[3]{8}}$$

$$=-\frac{1}{3}8^{\left(\frac{1}{3}-1\right)}\cdot\frac{1}{2\cdot 2}=\frac{-1}{48}$$

6. (b) : $\lim_{x\to\infty}\frac{n\log x}{[x]}-1=-1+\lim_{x\to\infty} n.\frac{\frac{\log x}{x}}{\frac{[x]}{x}}$

$$=-1+\frac{n(0)}{1}=-1$$

7. (c) : $\lim_{x\to 2^+}\frac{\{x\}}{(x-2)}\frac{\sin(x-2)}{(x-2)}$

$$= \lim_{x\to 2^+} \frac{\{x\}}{x-2} = \lim_{x\to 2^+} \frac{\{x\}}{\{x\}} =$$

8. (c) : $\lim_{x\to 0} \dfrac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x}$

$$= \lim_{x\to 0} \frac{\frac{a^{\tan x}-1}{\tan x}\tan x - \frac{a^{\sin x}-1}{\sin x}\sin x}{\tan x - \sin x}$$

$$= \lim_{x\to 0} \frac{\log_e a \tan x - \log_e a.\sin x}{\tan x - \sin x} = \log_e a$$

9. (c) : $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{3}\left(f(x+1) + \frac{5}{f(x+2)}\right)$

$$\lim_{x\to\infty} f(x) = \frac{1}{3}\left(f(\infty+1) + \frac{5}{f(\infty+2)}\right)$$

$$= \frac{1}{3}\left(f(\infty) + \frac{5}{f(\infty)}\right)$$

$\Rightarrow \quad K = \frac{1}{3}\left(K + \frac{5}{K}\right)$, then solve K.

10. (d) : I and II are false. The function $f(x) = 1/x$, $0 < x < 1$, is a counter example.
Statement III is true. Apply the intermediate value theorem to f on the closed interval $[a_1, b_1]$

11. (c) : $f(x) = \begin{cases} 1, & \text{if } x \in I \\ 0, & \text{if } x \notin I \end{cases}$ as $\sin \pi x = 0 \ \forall x \in I$

$\Rightarrow \quad f$ is discontinuous for all $x \in I$

12. (c) : $f'(0) = \underset{h\to 0}{\text{Lim}} \dfrac{e^{-1/h^2}}{h} = \underset{h\to 0}{\text{Lim}} \dfrac{1/h}{e^{1/h^2}}$

$$= \underset{h\to 0}{\text{Lim}} \frac{-1/h^2}{-e^{1/h^2}\cdot\frac{2}{h^3}} = \lim_{h\to 0} \frac{h^3}{2h^2 e^{1/h^2}} = 0$$

[Using L'Hospital rule]

Hence, f is differentiable at $x = 0$.

Also, $\lim_{x\to\pm\infty} e^{-\frac{1}{x^2}} \to 1 \Rightarrow$ (c) option is correct.

13. (d) : $g'(x) = f'(\tan^2 x - 2\tan x + 4)2\sec^2 x(\tan x - 1)$

Since, $f''(x) > 0 \Rightarrow f'(x)$ is increasing
$\tan^2 x - 2\tan x + 4 = (\tan x - 1)^2 + 3$
$\Rightarrow \quad \tan^2 x - 2\tan x + 4 \geq 3$
$\Rightarrow \quad f'(\tan^2 x - 2\tan x + 4) \geq f'(3) = 0$
(Given, $f'(3) = 0$)
Hence, $g'(x) > 0$ iff $\tan x - 1 > 0 \Rightarrow x \in (\pi/4, \pi/2)$

14. (c) : Replace x by $-x$
$f(2x^2 - 1) = -2x^3 f(-x)$
$\Rightarrow \quad 2x^3 f(x) = -2x^3 f(-x)$
$\Rightarrow \quad f(-x) = -f(x)$
$\Rightarrow \quad f(x)$ is an odd function and $f^{(n)}(0) = 0$, when n is even.

15. (b) : $g'(x) = x[f'(2x^2 - 1) - f'(1 - x^2)]$
$g'(x) > 0$ when both expressions are positive or both negative

Case-I : $x > 0 \Rightarrow f'(2x^2 - 1) > f'(1 - x^2)$
$\Rightarrow \quad 2x^2 - 1 > (1 - x^2)$ ($\because$ f' is increasing)

$\therefore \quad x \in \left(\sqrt{\frac{2}{3}}, \infty\right)$

Case-II : $x < 0 \Rightarrow f'(2x^2 - 1) < f'(1 - x^2)$

$\Rightarrow \quad 2x^2 - 1 < 1 - x^2 \quad \therefore \quad x \in \left(-\sqrt{\frac{2}{3}}, 0\right)$

16. (d) : Around $x = 2$, $\sin(|x| - 1) = \sin(x - 1)$
and $\sin(x - 1) > 0 \Rightarrow |\sin(x - 1)| = \sin(x - 1)$
we know, $\sin(x - 1) < 2$
Hence $|\sin(x - 1) - 2| = 2 - \sin(x - 1)$
$f(x) = 2 - \sin(x - 1)$ for $x \in (2 - \delta, 2 + \delta)$
$f'(2) = -\cos 1$

17. (c) : $f(x)$ is non-differentiable, when $p \sin x$ is integer
$p \sin x \leq p$
$p \sin x = r$, $r \in \{1, 2, ..., (p - 1)\}$ has two solutions for each 'r'
$p \sin x = p$ has only one solution in $x \in (0, \pi)$
$\therefore \quad$ the number $= 2(p - 1) + 1 = (2p - 1)$

18. (b) : $f'(x) - g'(x) =$ constant
[$\because \quad f''(x) = g''(x)$]

$f'(1) - g'(1) = -2$ as $f'(1) = 2$ and $g'(x) = 4$
$\Rightarrow \quad f'(x) - g'(x) = -2$
$\Rightarrow \quad (f - g)'(x) = -2x + k$
$f(2) - g(2) = k - 4 = 3 - 9 = -6 \Rightarrow k = -2$
$f(4) - g(4) = k - 8 = -10$

19. (d) : As 'x' approaches to zero

$\therefore \quad \sin x^2 \approx x^2$

$$\text{Limit} = \lim_{x\to 0}\frac{2f'(x)-6f'(2x)+4f'(4x)}{2x}$$

using L. H. rule

$$= \lim_{x\to 0}\frac{2f''(x)-12f''(2x)+16f'''(4x)}{2}$$

$$= 3f''(0) = 12 \qquad [\because \quad f''(0) = 4]$$

20. (c) : $f^{-1}(x) = \begin{cases} \sqrt{x-1} & , x \in [1,\infty) \\ 0 & , x = 0 \\ -\sqrt{-x-1} & , x \in (-\infty, -1] \end{cases}$

Clearly, f^{-1} is continuous on $[1, \infty)$ and on $(-\infty, -1]$, and as $x = 0$ is isolated point of its domain, the function is continuous at that point as well.

21. (b) : $f'(0) = \lim_{x\to 0}\frac{g(x)}{|x|} = \lim_{x\to 0}\frac{g'(x)}{|1|} = 0$

22. (b, c) : $\lim_{x\to\infty}\sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right)$

$$= \lim_{x\to\infty}\sqrt{x}\frac{(x+1-x)}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{\sqrt{1+0}+\sqrt{1}}$$

(Using L' Hospital rule)

(a) $\lim_{x\to 0}\frac{\log(1+x)-x}{x^2} = \lim_{x\to 0}\frac{\frac{1}{1+x}-1}{2x}$

$$= \lim_{x\to 0}\frac{-\frac{1}{(1+x)^2}}{2} = -\frac{1}{2}$$

(b) $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$

(c) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x\sqrt{x}} = \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$

(d) $\lim_{x\to 0}\frac{1}{\sqrt{x+\sqrt{x^2+2x}}} = \lim_{x\to 0}\frac{1}{\sqrt{1+\sqrt{1+\frac{2}{x}}}} = \frac{1}{\sqrt{2}}$

23. (a, b):

(a) $\lim_{x\to 0^+} x\left(\frac{e^{|x|+[x]}-2}{|x|+[x]}\right) = \lim_{x\to 0^+} x\left(\frac{e^{x+0}-2}{x+0}\right)$

$$= \lim_{x\to 0}(e^x - 2) = 1 - 2 = -1$$

(b) $\lim_{x\to 0-} x\left(\frac{e^{|x|+[x]}-2}{|x|+[x]}\right) = \lim_{x\to 0-} x\left(\frac{e^{-x-1}-2}{-x-1}\right) = 0$

24. (a, d) : $\lim_{x\to 1^+} f(x) = \lim_{x\to 1}\frac{1-1}{x^2-1} = 0$

For $\begin{cases} 1 < x < \sqrt{2} & , [x^2] = 1 \\ 0 < x < 1 & , [x^2] = 0 \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}\frac{0-1}{x^2-1} = \frac{-1}{0} = \infty$

$\Rightarrow \quad \lim_{x\to 1} f(x)$ does not exist.

25. (a, c) : Let $f(x) = \frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$

L.H.L. and R.H.L. are not equal at $x = 0$

$\therefore \quad \lim_{x\to 0} g(x)\left[\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right]$ exist when $g(x) = x$ or x^2

26. (a, b, c) :

$$\text{Lt}_{x\to 0}\frac{a\left(1+x+\frac{x^2}{2!}+.....\right)+b\left(1-\frac{x^2}{2!}.....\right)+c\left(1-x+\frac{x^2}{2!}\right)}{x^2} = 4$$

$\Rightarrow \quad a + b + c = 0, a - c = 0 \Rightarrow a = c$

$\frac{a-b+c}{2} = 4 \Rightarrow a - b + c = 8 \Rightarrow 2b = -8 \Rightarrow b = -4$

27. (a, b, c, d) :

$$L_1 = \lim_{x\to\alpha}\frac{1-\cos a(x-\alpha)(x-\beta)}{(x-\alpha)^2} = \frac{a^2(\alpha-\beta)^2}{2}$$

$$L_2 = \frac{a^2(\beta-\alpha)^2}{2}, L_3 = \frac{a^2}{2}$$

28. (b, d) : $\lim_{x\to\infty} 4\times\left(\tan^{-1}\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}\right) = y^2 + 4y + 5$

$$\Rightarrow \lim_{x\to\infty} 4\times\left(\frac{\tan^{-1}\frac{1}{2x+3}}{\frac{1}{2x+3}}\right)\frac{1}{2x+3} = y^2 + 4y + 5$$

$\Rightarrow\ y^2+4y+5=2\ \Rightarrow\ y=-1,-3$

29. (b, c) :

$$\lim_{x\to0^+} f(x)=\lim_{x\to0^+}\frac{\tan^2 x}{(x^2-[x]^2)}=\lim_{x\to0^+}\frac{\tan^2 x}{x^2}=1$$

As $x\to0^+$, $[x]=0$

Also, $\lim\limits_{x\to0^-} f(x)=\lim\limits_{x\to0^-}\sqrt{(x-[x])\cot(x-[x])}$

$$=\sqrt{\cot 1}$$

$$\cot^{-1}\left(\lim_{x\to0^-} f(x)\right)^2=\cot^{-1}(\cot 1)=1$$

30. (a, c) :

$$\lim_{x\to0^+} f(x)=\lim_{x\to0^+}[\tan x]+\sqrt{\tan x-[\tan x]}=0=f(x)$$

$\therefore$ $f(x)$ is continuous at $x=0$

$$\lim_{x\to\frac{\pi}{4}^-} f(x)=1=\lim_{x\to\frac{\pi}{4}^+} f(x)=f\left(\frac{\pi}{4}\right)$$

$\therefore$ $f(x)$ is continuous at $x=\dfrac{\pi}{4}$

it is clear that $f(x)$ is continuous at everywhere in $\left[0,\dfrac{\pi}{2}\right)$

31. (a, b, c) : $f(x)=\sec 2x+\operatorname{cosec} 2x$

$$=\frac{\sin 2x+\cos 2x}{\sin 2x\cos 2x}=\frac{2(\sin 2x+\cos 2x)}{\sin 4x}$$

Clearly it is discontinuous, when $4x=n\pi$ or $x=\dfrac{n\pi}{4}$ Option (a) and (b) also satisfy the condition since they are subsets of option (c).

32. (b, c, d) : (a) $f(x)$ is continuous at $x=1$

(b) $g(1^+)=0, g(1^-)=1\ \Rightarrow\ g(x)$ is discontinuous at $x=1$

(c) $\left.\begin{matrix}h(1^+)=1\\ h(1^-)=0\end{matrix}\right\}\Rightarrow h(x)$ is discontinuous at $x=1$

(d) L.H.L. $\neq$ R.H.L.

$\Rightarrow$ $\phi(x)$ is discontinuous at $x=1$

33. (a, b, c, d) :

$$f(x)=\begin{cases}1, & x \text{ is rational}\\ 0, & x \text{ is irrational}\end{cases},\ g(x)=\begin{cases}0, & x\in Q\\ 1, & x\in R-Q\end{cases}$$

$(f-g)(x)=1\ \ \forall\ x\in R$

(i) When $x\in Q$ say $x=p/q$, then $x!\ x$ is a multiple of π and $\cos^2 n!\ (x\pi)=1$.

$$\therefore\ \lim_{m\to\infty}\cos^{2m}(x!x\pi)=\lim_{m\to\infty}1^m=1\ \therefore\ f(x)=1$$

(ii) When x is irrational when $n!\,x\pi$ is not a multiple of π.

$\cos^2 n!\ (x\pi)\neq 1\ \ \therefore\ 0\le\cos^2 n!x\pi<1$

$$\therefore\ \lim_{m\to\infty}\cos^{2m}x=\lim_{m\to\infty}(\cos^2 n!\pi x)^m=0$$

34. (a, b, c) :

$$f(x)=\left.\begin{cases}\cos x+x, & x>0\\ \cos x-x, & x<0\end{cases}\right\}\begin{matrix}\text{L.H.D}=1\\ \text{R.H.D}=-1\end{matrix}\Big\}\text{L.H.D.}\neq\text{R.H.D.}$$

$$g(x)=\left.\begin{cases}\cos x-x, & x>0\\ \cos x+x, & x<0\end{cases}\right\}\begin{matrix}\text{L.H.D}=-1\\ \text{R.H.D}=1\end{matrix}\Big\}\text{L.H.D.}\neq\text{R.H.D.}$$

$$h(x)=\left.\begin{cases}\sin x+x, & x>0\\ -\sin x-x, & x<0\end{cases}\right\}\begin{matrix}\text{L.H.D}=2\\ \text{R.H.D}=-2\end{matrix}\Big\}\text{L.H.D.}\neq\text{R.H.D.}$$

$$\phi(x)=\left.\begin{cases}\sin x-x, & x>0\\ -\sin x+x, & x<0\end{cases}\right\}\begin{matrix}\text{L.H.D}=0\\ \text{R.H.D}=0\end{matrix}\Big\}\text{L.H.D.}=\text{R.H.D.}$$

35. (a, b, c): $f(x)=\begin{cases}x-3, & \text{if } x\ge 3\\ 3-x, & \text{if } 1\le x<3\\ \dfrac{x^2}{4}-\dfrac{3x}{2}+\dfrac{13}{4}, & \text{if } x<1\end{cases}$

$$f'(1^+)=\lim_{h\to0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to0}\frac{3-(1+h)-2}{h}=-1$$

$$f'(1^-)=\lim_{h\to0}\frac{\frac{(1-h)^2}{4}-\frac{3}{2}(1-h)+\frac{13}{4}-2}{-h}$$

$$=\lim_{h\to0}\frac{h^2-2h+6h}{-4h}=-1$$

$\Rightarrow$ f is continuous at $x=1$

36. (a, b, c, d) : $[|x|]-|[x]|=\begin{cases}0, & x=1\\ -1, & -1<x<0\\ 0, & 0\le x\le 1\\ 0, & 1<x\le 2\end{cases}$

$\Rightarrow$ range is $\{0,-1\}$

37. (b) : $\lim\limits_{x\to 0}\dfrac{|x|}{\sqrt{x^4+4x^2+7}}$

R.H.L. $= \lim\limits_{x\to 0^+}\dfrac{|x|}{\sqrt{x^4+4x^2+7}}$

$= \lim\limits_{x\to 0^+}\dfrac{x}{\sqrt{x^4+4x^2+7}} = \dfrac{0}{\sqrt{7}} = 0$

L.H.L. $= \lim\limits_{x\to 0}\dfrac{|x|}{\sqrt{x^4+4x^2+7}}$

$= \lim\limits_{x\to 0}\dfrac{-x}{\sqrt{x^4+4x^2+7}} = 0$

38. (a) : As $e^{1/x} - e^{-1/x} < e^{1/x} + e^{-1/x}$, result follows :

39. (c) : $\dfrac{1}{x^3} - 1 \le \left[\dfrac{1}{x^3}\right] \le \dfrac{1}{x^3}$

$\Rightarrow x^5\left(\dfrac{1}{x^3}-1\right) \le x^5\left[\dfrac{1}{x^3}\right] \le x^2, \ \forall\ x > 0$

and $x^2 \le x^5\left[\dfrac{1}{x^3}\right] \le x^5\left(\dfrac{1}{x^3}-1\right) \ \forall\ x < 0$

$\therefore \lim\limits_{x\to 0} x^5\left[\dfrac{1}{x^3}\right] = 0$

40. (d) : For any $x \neq 0$, $-1 \le \sin\dfrac{1}{x} \le 1$, but as $x \to 0$ $\sin\dfrac{1}{x}$ does not approach any particular value but oscillates between –1 and 1

41. (a) : $\lim\limits_{x\to 1^+} f(x) = 0 = \lim\limits_{x\to 1^-} f(x)$

$\therefore$ $f(x)$ is continuous at $x = 0$

42. (a) : As $x < 1$, $\lim\limits_{n\to\infty} x^{2n} = 0$

So, $\lim\limits_{x\to 1^-} f(x) = \dfrac{\log(2+1)-0}{1+0} = \log 3$

As $x > 1$, $\lim\limits_{x\to\infty} x^{2n} = \infty$

So, $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \dfrac{\log(2+x) - \sin x}{\dfrac{x^{2n}}{\dfrac{1}{x^{2n}}+1}} = -\sin 1$

Also, $f(1) = \dfrac{\log 3 - \sin 1}{2}$

Clearly, $f(x)$ does not have removable discontinuity at $x = 1$.

43. (c) : By given fact : $\dfrac{x\sin|x|}{1-|x|^2}$ is differentiable '0' but it is certainly not continuous at $x = 1$ and $x = -1$

$\Rightarrow$ not differentiable at $x = 1$ and $x = -1$

44. (a) : $f(x) = (x-3)\,|(x-3)(x-4)| + \cos(x-3)$ $[\because \cos|x-3| = \cos(x-3)]$

It is evident that $f(x)$ is not differentiable at $x = 4$

45. (b) : $h(x) = \sin x\,|\sin x|$ as $f(x) = |x|$ and $g(x) = \sin x$

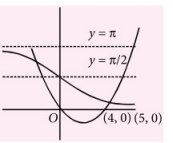
$f(g(x)) = f(\sin x) = |\sin x|$

Clearly, $h(x)$ is differentiable everywhere.

As $h(x)$ is the product of two continuous differentiable functions.

46. A → r; B → p; C → s; D → q

(A)



From the graph it is clear that $x = 5$ is the least positive integral value for which $x^2 - 4x > \cot^{-1}x$.

(B) $f'(x) = 2e^x + ae^{-x} + 2a + 1$

$= e^{-x}(2e^{2x} + (2a+1)e^x + a)$

$= 2e^{-x}(e^{2x} + (a+1/2)e^x + a/2)$

$= 2e^{-x}(e^x + a)(e^x + 1/2)$

From $f(x)$ to be increasing $f'(x) \ge 0\ \forall\ x \in R$

or $e^x + a \ge 0\ \forall\ x \in R$ or $a \ge 0$

(C) $\lim\limits_{x\to 0}\left(\dfrac{f^2(a+x)}{f(a)}\right)^{\frac{1}{x}}$

$= e^{\lim\limits_{x\to 0}\left(\frac{f^2(a+x)-f(a)}{xf(a)}\right)} = e^{\lim\limits_{x\to 0}\frac{f^2(a+x)-(f(a))^2}{x}} = e^4$

$(\because f(a) = 1)$

$\therefore \ k = 4$

(D) $\sin^{-1}(-x^2+4x-3) = \cos^{-1}(2-|x|)$

$\Rightarrow 0 \le -x^2+4x-3 \le 1 \Rightarrow x \in [1, 3]$ ...(i)

Also, $0 \le 2 - |x| \le 1$

$\Rightarrow x \in [-2, -1] \cup [1, 2]$ ...(ii)

from (i) and (ii), $x \in \{1, 2\}$

47. A → r, s; B → p, q; C → p, q; D → p, r

(A) The given function is clearly continuous at all points except possibly at $x = \pm 1$

As $f(x)$ is an even function, so we need to check its continuity only at $x = 1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to 1^-} (ax^2 + b) = \lim_{x\to 1^+} \frac{1}{|x|} \Rightarrow a + b = 1$$

or $a = 1 - b$ ...(1)

Clearly, $f(x)$ is differentiable for all x, except possibly at $x = \pm 1$. As $f(x)$ is even function, so we need to check its differentiability at $x = 1$ only.

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$\Rightarrow \lim_{x\to 1} \frac{ax^2 + b - 1}{x - 1} = \lim_{x\to 1^+} \frac{\frac{1}{|x|} - 1}{x - 1}$$

$$\Rightarrow \lim_{x\to 1} \frac{ax^2 - a}{x - 1} = \lim_{x\to 1} \frac{-1}{1}$$

$$\Rightarrow 2a = -1 \Rightarrow a = \frac{-1}{2} \text{ and } b = -\frac{1}{2} \quad \text{[from (1)]}$$

(B) $f(x) = \text{sgn}\,(x^2 - ax + 1)$ is discontinuous then $x^2 - ax + 1 = 0$ must have only one real root. Hence $a = \pm 2$.

(C) $f(x) = [2 + 3|n| \sin x], n \in N$ has exactly 11 points of discontinuity in $x \in (0, \pi)$

$2(3)|n| - 1 = 11 \Rightarrow 6|n| = 12 \Rightarrow n = \pm 2$

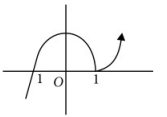
($\because$ $f(x) = [a + b\sin x]$ $0 < x < \pi$ $\forall$ $a \in I$ has $2b - 1$ points of non-differentiability)

(D) If $f(x) = |||x| - 2| + a|$ has exactly three points of non-differentiability.

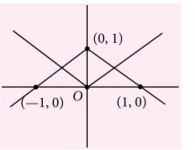
$f(x)$ is non-differentiable at $x = 0$, $|x| - 2 = 0$ or $x = 0, \pm 2$. Hence, the value of a must be positive, as negative value of a allows $||x| - 2| + a = 0$ to have real roots, which gives two more points of non-differentiability.

48. A → p, q; B → p, s; C → r, s; D → p, q

(A) $f(x) = (x + 1)|x - 1| = \begin{cases} x^2 - 1; & x \geq 1 \\ 1 - x^2; & x < 1 \end{cases}$



(B) $f(x) = \min\{|x|, 1 - |x|\}$



(C) $f(x) = \{x\} + 2[x] = x + [x]$ at $x = I$

L.H.L. $= I + (I - 1) = 2I - 1$

R.H.L. $= I + I = 2I = f(I)$.

$\therefore$ Not continuous hence not differentiable at integral points but increasing.

(D) $f(x) = \sqrt{\cos^2 \frac{\pi x}{2}} = \left|\cos \frac{\pi x}{2}\right|$ is continuous and differentiable

49. (7) :

$$\lim_{x\to 0} \frac{\cos 4x + a\cos 2x + b}{x^4} \Rightarrow \frac{0}{0} \text{ if } 1 + a + b = 0$$

$$= \lim_{x\to 0} \frac{-4\sin 4x - 2a\sin 2x}{4x^3} \to \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-16\cos 4x - 4a\cos 2x}{12x^2} \Rightarrow \frac{0}{0}$$

if $-16 - 4a = 0 \Rightarrow a = -4$

and $1 + a + b = 0 \Rightarrow b = 3$

$\therefore$ $b - a = 7$

50. (6) : $\lim_{x\to 0} \frac{2f(x) - 3f(2x) + f(4x)}{x^2} \to \frac{0}{0}$ form

$$= \lim_{x\to 0} \frac{2f'(x) - 6f'(2x) + 4f'(4x)}{2x} \to \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0} \frac{2f''(x) - 12f''(2x) + 16f''(4x)}{2}$$

$$= \frac{(2 - 12 + 16)2}{2} = 6$$

51. (2) :

$$f(x)=\lim_{n\to\infty}\left\{\frac{(1-x)(1+x)(1+x^2)(1+x^4).....n\text{ factors}}{1-x}\right\}$$

as $|x|<1 \ \therefore\ x^{2n}\to 0$ as $n\to\infty$

$$\therefore f(x)=\lim_{n\to\infty}\frac{1-x^{2n}}{1-x}\therefore f\left(\frac{1}{2}\right)=\frac{1-0}{1-\frac{1}{2}}\therefore f\left(\frac{1}{2}\right)=2$$

52. (6) : $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)^2}{h}$

$$=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)[1+hg(h)H(h)-1]}{h}$$

$$=f(x)\lim_{h\to 0}g(h)\cdot H(h)=6f(x)\ =Kf(x)\ \therefore\ K=6$$

53. (2) : $\lim_{x\to 0}\dfrac{e^{\tan x}-e^{\sin x}}{\tan x-\sin x}\cdot\dfrac{\tan x}{e^{x/32}-1}\dfrac{(1-\cos x)}{(1-\cos 4x)}$

$$=1.\frac{1}{\frac{1}{32}}\times\frac{1}{16}=2$$

54. (2) :

$$\lim_{x\to 0}\frac{1-\cos^n(1-\cos x)}{(1-\cos x)^2}\cdot\left(\frac{1-\cos x}{x^2}\right)^2\cdot\frac{x^4}{\tan^m x}=1$$

$$\Rightarrow m=4 \text{ and } \frac{n}{2}.\frac{1}{4}=1\Rightarrow n=8$$

55. (1) : Here $f(x)$ must be $3\tan^{-1}x$

$f(\tan(1/3))=3\tan^{-1}(\tan 1/3)=1$

56. (4) : $y'(2\sqrt{x})=e^{\sqrt{x}}-e^{-\sqrt{x}}$

$$\Rightarrow\ y''(2\sqrt{x})+2y'\frac{1}{2\sqrt{x}}=\frac{e^x+e^{-\sqrt{x}}}{2\sqrt{x}}$$

$$\Rightarrow\ 4xy''+2y'-y=0$$

$$\therefore\ k=4$$

57. (6) : $\dfrac{dy}{dx}=\dfrac{3(1-t^2)}{2t}\div\dfrac{dx}{dt}=\dfrac{-(t^2+1)}{6t^3}$

putting this, we get $k=36$

58. (8) : $\dfrac{x^4}{(x-1)(x-2)}=x^2+3x+7+\dfrac{16}{x-2}-\dfrac{1}{x-1}$

Third order derivative $=0+\dfrac{16(-3!)}{(x-2)^4}+\dfrac{3!}{(x-1)^4}$

$$\Rightarrow\ -12k=-96$$

$$\Rightarrow\ k=8$$

59. (1) : $\dfrac{dy}{dx}=\dfrac{2-2x}{2\sqrt{2x-x^2}}=\dfrac{1-x}{y}$ ...(i)

$$\Rightarrow\ y\frac{dy}{dx}=1-x\text{, Differentiating again}$$

$$\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}=-1$$

$$\Rightarrow\ y^3\frac{d^2y}{dx^2}=-y^2-(1-x)^2$$

$$=-2x+x^2-(1-x)^2=-1\quad\text{[From (i)]}$$

$$\therefore\ k=1$$

60. (2) : RHL $=\lim_{x\to -1^+}\dfrac{\pi-\cos^{-1}(x)}{\sqrt{x+1}}\cdot\dfrac{1}{\sqrt{\pi}+\sqrt{\cos^{-1}x}}$

$$=\lim_{x\to -1^+}\frac{\cos^{-1}(-x)}{\sqrt{x+1}}\cdot\frac{1}{\sqrt{\pi}+\sqrt{\cos^{-1}x}}$$

put $\cos^{-1}(-x)=t$

$$=\sqrt{2}\cdot\frac{1}{\sqrt{\pi}+\sqrt{\pi}}=\frac{1}{\sqrt{2\pi}}$$

$$\therefore\ \lambda=2$$

61. (1) : $f=g^{-1}$ *i.e.* $g^{-1}(x)=f(x)$

where $g(x)=y=2\tan^{-1}(e^x)-\dfrac{\pi}{2}$

$$\Rightarrow\tan^{-1}(e^x)=\frac{\pi}{4}+\frac{y}{2}$$

$$\Rightarrow x=\log\tan\left(\frac{\pi}{4}+\frac{y}{2}\right)$$

$$\Rightarrow g^{-1}(y)=\log\left(\frac{\pi}{4}+\frac{y}{2}\right)$$

$$\Rightarrow g^{-1}(x)=f(x)=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$\Rightarrow f'(x)=\frac{1}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}\cdot\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)\times\frac{1}{2}$$

$$f'(0)=\frac{1}{\tan\frac{\pi}{4}}\times\sec^2\frac{\pi}{4}\times\frac{1}{2}=1$$

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Continuity and Differentiability

CONTINUITY

	Continuity of a Function	Definitions
1.	At a point	A real valued function $f(x)$ is continuous at a point $x = a$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.
2.	In an interval	A real valued function $f(x)$ is said to be continuous in an interval (a, b) if it is continuous at every point of the interval. A function which is continuous at entire real line $(-\infty, \infty)$ is said to be continuous everywhere.

Discontinuity of a Function

1.	At a point	A real valued function $f(x)$ is said to be discontinuous at $x = a$, if it is not continuous at $x = a$. The discontinuity may be due to any of the following reasons: (i) $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ or both may not exist. (ii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are not equal. (iii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are equal but both may not be equal to $f(a)$.
2.	In an interval	A real valued function $f(x)$ is said to be discontinuous if it is not continuous in atleast one point in the given interval.

Types of Discontinuity

1.	Removable discontinuity	$f(x)$ is said to have a removable discontinuity at $x = a$ if $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a^+} f(x) \neq f(a)$ or if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$
2.	Non-removable discontinuity	(i) First kind: If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ (ii) Second kind: If $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ or both do not exist.

- **Algebra of Continuous Functions**

(A) If f and g are two real functions continuous at a real number c, then

(i) $f + g$, $f - g$ and fg are continuous, at $x = c$

(ii) $\frac{1}{g}$ and $\frac{f}{g}$ is continuous at $x = c$, $g(c) \neq 0$.

(B) Let f and g be real valued functions such that fog is defined. If g is continuous at $x = a$ and f is continuous at $g(a)$, then fog is continuous at $x = a$.

DIFFERENTIABILITY

Differentiability of a function		Definitions
At a point		A real valued function $f(x)$, is said to be differentiable at a point $x = a$ if and only if $\lim_{\substack{x\to a-h \\ h\to 0}} \frac{f(a-h)-f(a)}{-h} = \lim_{\substack{x\to a+h \\ h\to 0}} \frac{f(a+h)-f(a)}{h}$ or $Lf'(a) = Rf'(a)$
1.	Left Hand Derivative (L.H.D.)	If $y = f(x)$ is a real valued function defined on (a, b) and 'c' is any real number, then $Lf'(c) = \lim_{\substack{x\to c-h \\ h\to 0}} \frac{f(c-h)-f(c)}{-h}$
2.	Right Hand Derivative (R.H.D.)	If $y = f(x)$ is a real valued function defined on (a, b) and 'c' is any real number, then $Rf'(c) = \lim_{\substack{x\to c+h \\ h\to 0}} \frac{f(c+h)-f(c)}{h}$

- A real valued function $f(x)$ is not differentiable due to following reasons:
  (i) If $Lf'(a) \neq Rf'(a)$, then function is not differentiable at $x = a$.
  (ii) If $f(x)$ is not differentiable at atleast one point in an interval, then it is not differentiable in that interval.
- **Relation between Continuity and Differentiability**
  If a function is differentiable at a point, it is necessarily continuous at that point. But the converse is not always true.
- **Algebra of Differentiable Functions**
  (A) Sum, difference, product and division of two differentiable functions are differentiable.
  (B) The composite of two differentiable functions is also differentiable.

Derivative of Functions

1.	Sum or Difference	$(u \pm v)' = u' \pm v'$
2.	Product Rule	$(uv)' = u'v + uv'$
3.	Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, $v \neq 0$
4.	Composite Function (Chain Rule)	(a) Let $y = f(t)$ and $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ (b) Let $y = f(t)$, $t = g(u)$ and $u = m(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$

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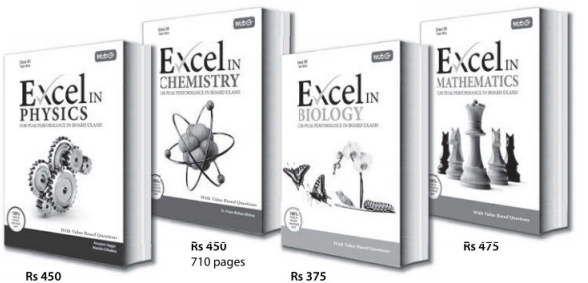

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5.	Parametric Function	If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$, $f'(t) \neq 0$
6.	Second Order Derivative	Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ If $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$ or $\frac{d^2y}{dx^2} = f''(x)$
7.	Logarithmic Function	If $y = u^v$, where u and v are the functions of x, then $\log y = v \log u$. Differentiating w.r.t. x, we get $\frac{d}{dx}(u^v) = u^v\left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right]$
8.	Implicit Function	Here, we differentiate the function of type $f(x, y) = 0$.

Some General Derivatives

Function	Derivative		Derivative		Derivative
x^n	nx^{n-1}	$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\operatorname{cosec}^2 x$	$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	e^{ax}	ae^{ax}	e^x	e^x
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}};\ x \in (-1,1)$	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}};x \in (-1,1)$	$\tan^{-1} x$	$\frac{1}{1+x^2};\ x \in R$
$\cot^{-1} x$	$-\frac{1}{1+x^2};\ x \in R$	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}};x \in R - [-1,1]$	$\operatorname{cosec}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}};$ $x \in R-[-1,1]$
$\log_e x$	$\frac{1}{x};\ x > 0$	a^x	$a^x \log_e a;\ a > 0$	$\log_a x$	$\frac{1}{x \log_e a};x>0$ and $a>0$,

Mean Value Theorems

1.	Rolle's Theorem	If $f(x)$ is a real valued function defined on $[a, b]$ such that it is (i) continuous on $[a, b]$ (ii) differentiable on (a, b) and (iii) $f(a) = f(b)$, then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$
	Geometrical meaning	The tangent at point c on the curve $y = f(x)$ is parallel to x-axis.
2.	Lagrange's Mean Value Theorem	If $f(x)$ is defined on $[a, b]$ such that it is (i) continuous on $[a, b]$ and (ii) differentiable on (a, b), then there exists at least one $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$
	Geometrical meaning	The tangent at point c on the curve $y = f(x)$ is parallel to the chord joining end points.

VERY SHORT ANSWER TYPE (1 MARK)

1. If $f(x) = x^2 + 2x + 7$, find $f'(3)$.

2. Prove that the function $f(x) = \dfrac{3x^2 - 7x + 1}{x^2 - 4}$ is continuous.

3. Differentiate $\sin(x^2+5)$.

4. Find 'c' of the mean value theorem for the function $f(x) = 2x^2 - 10x + 29$ in $[2, 7]$.

5. Find the value of the constant λ so that the given function is continuous at $x = -1$.

$$f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x+1}, & x \neq -1 \\ \lambda, & x = -1 \end{cases}$$

SHORT ANSWER TYPE (4 MARKS)

6. Show that $f(x) = \begin{cases} \dfrac{|x-a|}{x-a}, & x \neq a \\ 1, & x = a \end{cases}$

is discontinuous at $x = a$.

7. If $y = e^x(\sin x + \cos x)$, then prove that

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$

8. Examine the function f for continuity and differentiability at $x = 1$, where

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ \dfrac{1}{x}, & x > 1 \end{cases}$$

9. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x - y)$, then prove that

$$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}.$$

10. Verify Rolle's theorem for the function $f(x) = x^3 - 6x^2 + 11x - 6$ in the interval $[1, 3]$.

11. If $y = \log\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$, then show that

$$\frac{dy}{dx} - \sec x = 0.$$

LONG ANSWER TYPE (6 MARKS)

12. If $x^y + y^x = a^b$, find $\dfrac{dy}{dx}$.

13. Differentiate $(3x^4 - 2x + 6)^4 (x - 3)^{2/3}$

14. Let $f(x) = \begin{cases} \dfrac{1 - \cos 4x}{x^2}, & \text{if } x < 0 \\ a, & \text{if } x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}, & \text{if } x > 0 \end{cases}$

Determine the value of a so that $f(x)$ is continuous at $x = 0$.

15. Find a point on the parabola $y = (x - 3)^2$, where the tangent is parallel to the chord joining $(3, 0)$ and $(4, 1)$.

SOLUTIONS

1. We have, $f(x) = x^2 + 2x + 7$

$\Rightarrow f'(x) = 2x + 2$

So, $f'(3) = 2 \times 3 + 2 = 6 + 2 = 8$

2. $f(x) = \dfrac{3x^2 - 7x + 1}{x^2 - 4}$, so domain of $f = R - \{-2, 2\}$.

Since, a rational function is continuous at every point of its domain and here f is a rational function, therefore, f is continuous at every point of its domain.

$\Rightarrow$ f is a continuous function.

3. Let $y = \sin(x^2 + 5)$

Diff. w.r.t. x, we get

$$\frac{dy}{dx} = \cos(x^2 + 5)(2x + 0) = 2x\cos(x^2 + 5).$$

4. $f(x) = 2x^2 - 10x + 29 \Leftrightarrow f'(x) = 4x - 10$

Now, $f'(c) = \dfrac{f(7) - f(2)}{(7 - 2)} \Rightarrow 4c - 10 = \dfrac{57 - 17}{5}$

$\Rightarrow 4c = 18 \Rightarrow c = 4.5 \in]2, 7[$

5. Since $f(x)$ is continuous at $x = -1$. Therefore,

$$\lim_{x \to -1} f(x) = f(-1)$$

$$\Rightarrow \lim_{x \to -1} \frac{x^2 - 2x - 3}{x + 1} = \lambda$$

$$\Rightarrow \lim_{x \to -1} \frac{(x-3)(x+1)}{x+1} = \lambda$$

$$\Rightarrow \lim_{x \to -1} (x - 3) = \lambda \Rightarrow \lambda = -4$$

6. We have, $f(x)=\begin{cases}\dfrac{|x-a|}{x-a} & , x\neq a\\ 1 & , x=a\end{cases}$

$$\Rightarrow \quad f(x)=\begin{cases}\dfrac{-(x-a)}{x-a} & , x<a\\ \dfrac{(x-a)}{x-a} & , x>a\\ 1 & , x=a\end{cases}$$

$$\therefore \lim_{x\to a^-} f(x)=\lim_{x\to a^-}\frac{-(x-a)}{x-a}$$

$$=\lim_{x\to a^-}(-1)=-1$$

and $\lim_{x\to a^+} f(x)=\lim_{x\to a^+}\frac{(x-a)}{x-a}=\lim_{x\to a^+}(1)=1.$

Thus, $\lim_{x\to a^-} f(x)\neq\lim_{x\to a^+} f(x)$

$\Rightarrow \lim_{x\to a} f(x)$ does not exist.

$\Rightarrow$ f is discontinuous at $x=a$.

7. We have, $y=e^x(\sin x+\cos x)$ ...(i)

Diff. (i) w.r.t x, we get

$$\frac{dy}{dx}=e^x\frac{d}{dx}(\sin x+\cos x)+(\sin x+\cos x)\frac{d}{dx}(e^x)$$

$=e^x(\cos x-\sin x)+(\sin x+\cos x)\cdot e^x$

$$\Rightarrow \frac{dy}{dx}=2e^x\cos x \qquad ...(ii)$$

Diff. (ii) w. r. t. x, we get

$$\frac{d^2y}{dx^2}=2\cdot\frac{d}{dx}(e^x\cos x)$$

$$\Rightarrow \frac{d^2y}{dx^2}=2\cdot\left\{e^x\cdot\frac{d}{dx}(\cos x)+\cos x\cdot\frac{d}{dx}(e^x)\right\}$$

$$=2\cdot\left\{e^x(-\sin x)+(\cos x)e^x\right\}$$

$$=2e^x(\cos x-\sin x)$$

Now, $\left(\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+2y\right)$

$=2e^x(\cos x-\sin x)-4e^x\cos x+2e^x(\sin x+\cos x)=0$

Hence, $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+2y=0.$

8. Given, $f(x)=\begin{cases}x^2, x\le 1\\ \dfrac{1}{x}, x>1\end{cases}\Rightarrow D_f=R$

$\Rightarrow$ f is defined in a neighbourhood of 1 and $f(1)=1^2=1$.

Now, $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} x^2=1^2=1$

Also, $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{1}{x}=\frac{1}{1}=1$

$$\Rightarrow \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$$

$\Rightarrow$ f is continuous at $x=1$.

Now, $Lf'(1)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$$=\lim_{h\to 0}\frac{(1-h)^2-1}{-h}$$

$$=\lim_{h\to 0}\frac{-2h+h^2}{-h}=\lim_{h\to 0}(2-h)=2-0=2$$

Also $Rf'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

$$=\lim_{h\to 0}\frac{\frac{1}{1+h}-1}{h}$$

$$=\lim_{h\to 0}\frac{1-(1+h)}{h(1+h)}=\lim_{h\to 0}\frac{-1}{1+h}=\frac{-1}{1+0}=-1$$

$\Rightarrow Lf'(1)\neq Rf'(1)$

$\Rightarrow f$ is not differentiable at $x=1$.

9. Let $x=\sin\theta$ and $y=\sin\phi$

$\Rightarrow \theta=\sin^{-1}x$ and $\phi=\sin^{-1}y$, then

$$\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$$

$$\Rightarrow \sqrt{1-\sin^2\theta}+\sqrt{1-\sin^2\phi}=a(\sin\theta-\sin\phi)$$

$$\Rightarrow \cos\theta+\cos\phi=a(\sin\theta-\sin\phi)$$

$$\Rightarrow 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)=a\cdot 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\Rightarrow \cos\left(\frac{\theta-\phi}{2}\right)=a\sin\left(\frac{\theta-\phi}{2}\right)\Rightarrow \cot\left(\frac{\theta-\phi}{2}\right)=a$$

$$\Rightarrow \frac{\theta-\phi}{2}=\cot^{-1}a \Rightarrow \theta-\phi=2\cot^{-1}a$$

$$\Rightarrow \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a \quad ...(i)$$

Differentiating (i) w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}.$$

10. Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[1, 3]$.

(ii) $f'(x)=3x^2-12x+11$, which clearly exists for all values of $x\in[1,3]$.

So, $f(x)$ is differentiable on the open interval $]1, 3[$.

(iii) $f(1)=(1^3-6\times1^2+11\times1-6)=0$

and $f(3)=(3^3-6\times3^2+11\times3-6)=0$

$\therefore \quad f(1)=f(3)$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist some $c\in\,]1,3[$ such that $f'(c)=0$

Now, $f'(c)=0 \Rightarrow 3c^2-12c+11=0$

$$\Rightarrow c=\frac{12\pm\sqrt{144-132}}{6}\Rightarrow c=\left(2\pm\frac{1}{\sqrt{3}}\right)$$

Clearly, both the values of c lie in the interval $]1, 3[$.

Hence, Rolle's theorem is verified.

11. Given, $y=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}\cdot\frac{d}{dx}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)$$

$$=\frac{1}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}\cdot\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)\cdot\frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$=\frac{\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)}\cdot\frac{1}{\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}\cdot\left(0+\frac{1}{2}.1\right)$$

$$=\frac{1}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}$$

$$=\frac{1}{\sin2\left(\frac{\pi}{4}+\frac{x}{2}\right)}=\frac{1}{\sin\left(\frac{\pi}{2}+x\right)}$$

$$=\frac{1}{\cos x}=\sec x$$

$$\Rightarrow \frac{dy}{dx}-\sec x=0 \text{ Hence proved.}$$

12. Let $u=x^y$ and $v=y^x$

Then, $u+v=a^b \Rightarrow \frac{du}{dx}+\frac{dv}{dx}=0$ ...(i)

$[\because\ a^b=\text{constant}]$

Now, $u=x^y$. Taking log on both the sides,

$\Rightarrow \log u=y\log x$ ...(ii)

Differentiating (ii) w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=y\cdot\frac{1}{x}+\log x\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx}=u\left[\frac{y}{x}+\log x\cdot\frac{dy}{dx}\right]$$

$$\Rightarrow \frac{du}{dx}=x^y\left[\frac{y+x\log x\cdot\frac{dy}{dx}}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{y-1}\left[y+x\log x\cdot\frac{dy}{dx}\right]$$

Also, $v=y^x$. Taking log on both sides,

$\Rightarrow \log v=x\log y$ ...(iii)

Differentiating (iii) w.r.t. x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx}=x\cdot\frac{1}{y}\cdot\frac{dy}{dx}+\log y$$

$$\Rightarrow \frac{dv}{dx}=v\left[\frac{x}{y}\cdot\frac{dy}{dx}+\log y\right]$$

$$\Rightarrow \frac{dv}{dx}=y^x\left\{\frac{x\cdot\frac{dy}{dx}+y\log y}{y}\right\}$$

$$\Rightarrow \frac{dv}{dx} = y^{x-1}\left\{x\cdot\frac{dy}{dx} + y\log y\right\}$$

From (i), we have $\frac{du}{dx} + \frac{dv}{dx} = 0$

$$\Rightarrow x^{y-1}\left\{y + x\log x\cdot\frac{dy}{dx}\right\} + y^{x-1}\left\{x\frac{dy}{dx} + y\log y\right\} = 0$$

$$\Rightarrow \{x^y(\log x) + x\cdot y^{x-1}\}\cdot\frac{dy}{dx} = -\{y\cdot x^{y-1} + y^x\log y\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\{y\cdot x^{y-1} + y^x(\log y)\}}{\{x^y(\log x) + xy^{x-1}\}}$$

13. Let $y = (3x^4 - 2x + 6)^4\,(x-3)^{2/3}$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = (3x^4 - 2x + 6)^4\cdot\frac{d}{dx}((x-3)^{2/3}) + (x-3)^{2/3}\cdot\frac{d}{dx}((3x^4 - 2x + 6)^4)$$

$$= (3x^4 - 2x + 6)^4\cdot\frac{2}{3}(x-3)^{-1/3}\cdot\frac{d}{dx}(x-3) + (x-3)^{2/3}\cdot 4(3x^4 - 2x + 6)^3\cdot\frac{d}{dx}(3x^4 - 2x + 6)$$

$$= \frac{2}{3}(3x^4 - 2x + 6)^4(x-3)^{-1/3} + 4(x-3)^{2/3}(3x^4 - 2x + 6)^3\cdot(3\cdot 4x^3 - 2\cdot 1 + 0)$$

$$= 2(3x^4 - 2x + 6)^3\times\left[\frac{3x^4 - 2x + 6}{3(x-3)^{1/3}} + 2(x-3)^{2/3}(12x^3 - 2)\right]$$

$$= 2(3x^4 - 2x + 6)^3\times\left(\frac{3x^4 - 2x + 6 + 12(x-3)(6x^3 - 1)}{3(x-3)^{1/3}}\right)$$

$$= \frac{2(3x^4 - 2x + 6)^3(75x^4 - 216x^3 - 14x + 42)}{3(x-3)^{1/3}}$$

14. For $f(x)$ to be continuous at $x = 0$, we must have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = a \quad ...(i)$$

Now,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\frac{1-\cos 4x}{x^2}$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0}\frac{2\sin^2 2x}{x^2}$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = 2\lim_{x\to 0}\left(\frac{\sin 2x}{x}\right)^2$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = 2\times 4\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 8(1)^2 = 8 \quad ...(ii)$$

and, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0}\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4}$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0}\frac{\sqrt{x}\left(\sqrt{16+\sqrt{x}} + 4\right)}{16 + \sqrt{x} - 16}.$$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0}(\sqrt{16+\sqrt{x}} + 4) = 4 + 4 = 8 \quad ...(iii)$$

From (i), (ii) and (iii), we get $a = 8$.

15. Let us apply Lagrange's mean-value theorem for the function $f(x) = (x-3)^2$ in the interval $[3, 4]$.

(i) $f(x)$ being a polynomial function, so it is continuous on $[3, 4]$.

(ii) $f'(x) = 2(x-3)$, which exists for all $x \in]3, 4[$

So, $f(x)$ is differentiable on $]3, 4[$

Thus, both the conditions of Lagrange's mean value theorem are satisfied.

So, there must exist a point $c \in]3, 4[$ such that

$$f'(c) = \frac{f(4) - f(3)}{(4-3)} = \frac{(1)^2 - (0)^2}{1} = 1$$

Now, $f'(c) = 1 \Rightarrow 2(c-3) = 1 \Rightarrow c = \frac{7}{2} \in]3, 4[$

Now, $x = \frac{7}{2}$ and $y = (x-3)^2 \Rightarrow y = \frac{1}{4}$.

Thus, at the point $\left(\frac{7}{2}, \frac{1}{4}\right)$ on the given curve, the tangent is parallel to the chord joining (3, 0) and (4, 1).

■■

SOLVED PAPER 2014

J&K CET

1. Find the value of $\cos\left(\frac{29\pi}{3}\right)$.

(a) 1 (b) 0 (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

2. Find the derivative of $\sqrt{2}x+2\sqrt{x}-\frac{1}{\sqrt{x}}$.

(a) $\sqrt{2}+\frac{1}{\sqrt{x}}\left(1-\frac{1}{2x}\right)$ (b) $\sqrt{2}-\frac{1}{\sqrt{x}}\left(1+\frac{1}{2x}\right)$

(c) $\sqrt{2}-\frac{1}{\sqrt{x}}\left(1-\frac{1}{2x}\right)$ (d) $\sqrt{2}+\frac{1}{\sqrt{x}}\left(1+\frac{1}{2x}\right)$

3. Solve $\frac{dy}{dx}+\frac{y}{x}=x^3$.

(a) $y=\frac{x^4}{5}+\frac{C}{x}$ (b) $y=\frac{x^3}{3}+Cx$

(c) $y=\frac{x^3}{3}+C$ (d) $y=\frac{x^3}{4}+x$

4. Given two vectors are $\hat{i}-\hat{j}$ and $\hat{i}+2\hat{j}$ the unit vector is coplanar with the two vectors and perpendicular to the first. Find the vector.

(a) $\pm\frac{1}{\sqrt{2}}(\hat{i}+\hat{k})$ (b) $\pm\frac{1}{\sqrt{5}}(2\hat{i}+\hat{j})$

(c) $\pm\frac{1}{\sqrt{2}}(\hat{i}+\hat{j})$ (d) $\pm\frac{1}{\sqrt{2}}(2\hat{i}+3\hat{j})$

5. Solve the linear programming problem : Max. $Z=x+2y$ subject to constraints : $x-y\le 10$, $2x+3y\le 20$, $x\ge 0$, $y\ge 0$

(a) Max. $Z=10$ (b) Max. $Z=30$

(c) Max. $Z=40$ (d) Max. $Z=50$

6. Find the intersection of the spheres $x^2+y^2+z^2+7x-2y-z=13$ and $x^2+y^2+z^2-3x+3y+4z=8$

(a) $x-y-z=1$ (b) $x-2y-z=1$

(c) $x-y-2z=1$ (d) $2x-y-z=1$

7. Convert $\frac{(i+1)}{\left(\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right)\right)}$ in polar form.

(a) $\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)$

(b) $\cos\left(\frac{\pi}{2}\right)-i\sin\left(\frac{\pi}{2}\right)$

(c) $\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$

(d) $\sqrt{2}\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)$

8. Solve $\sin^{-1}2x+\cos^{-1}2x+2\tan^{-1}x=\pi$.

(a) 1 (b) 0 (c) −1 (d) $\frac{1}{\sqrt{2}}$

9. For what values of x, the numbers -1, x, $-\frac{3}{4}$ are in Geometric progression?

(a) $\frac{3}{2}$ (b) $\frac{\sqrt{3}}{4}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{3}{\sqrt{2}}$

10. $(1+i)^3+(1-i)^3=?$

(a) 1 (b) −2 (c) 0 (d) −4

11. Find the length of the diagonal of the parallelopiped formed by planes drawn through the points (2, 3, 5) and (5, 9, 7), parallel to the co-ordinate planes.

(a) $\sqrt{38}$ units (b) 7 units
(c) $\sqrt{155}$ units (d) 13 units

12. If $g(x) = (x^2 + 2x + 3)f(x)$, $f(0) = 5$ and $\lim_{x\to 0}\left(\frac{f(x)-5}{x}\right) = 4$, then $f'(0)$ is equal to
(a) 22 (b) 18 (c) 20 (d) 25

13. Find $\int \frac{1}{\sqrt{7-x^2}}dx$
(a) $\frac{1}{2\sqrt{7}}\log\left|\frac{\sqrt{7}+x}{\sqrt{7}-x}\right| + c$
(b) $\sin^{-1}\left(\frac{x}{\sqrt{7}}\right) + c$
(c) $\log\left|x+\sqrt{x^2-7}\right| + c$
(d) $\frac{1}{2\sqrt{7}}\log\left|\frac{x-\sqrt{7}}{x+\sqrt{7}}\right| + c$

14. Find the range of the function $f: [0, 1] \to R$, $f(x) = x^3 - x^2 + 4x + 2\sin^{-1}x$.
(a) $[-(\pi + 2), 0]$ (b) $[0, 4 + \pi]$
(c) $[2, 3]$ (d) $(0, 2+\pi]$

15. Solve the inequality $3x + 2 > -16$, $2x - 3 \le 11$
(a) $(-6, 7)$ (b) $[-6, 7)$
(c) $(-6, 7]$ (d) $[-6, 7]$

16. Events A, B and C are mutually exclusive events such that $P(A) = (3x + 1)/3$, $P(B) = (1 - x)/4$ and $P(C) = (1 - 2x)/2$. Find the set of possible values of x in the interval.
(a) $\left[-\frac{1}{3}, \frac{1}{2}\right]$ (b) $\left[\frac{1}{3}, \frac{2}{3}\right]$
(c) $\left[\frac{1}{3}, \frac{13}{3}\right]$ (d) $[0, 1]$

17. If $\cos^{-1}\frac{(1-x^2)}{(1+x^2)} + \tan^{-1}x = \frac{\pi}{2}$, then x is equal to
(a) 0 (b) 1 (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

18. There are 7 horses in a race, Mr. X selected 2 horses at random and bet on them. The probability that Mr. X selected the winning horse, is
(a) $\frac{1}{7}$ (b) $\frac{4}{7}$ (c) $\frac{3}{7}$ (d) $\frac{2}{7}$

19. If $x^y = e^{(x-y)}$, then find $\frac{dy}{dx}$.
(a) $\frac{\log x}{(1+\log x)^2}$ (b) $\frac{(\log x)}{(1+\log x)}$
(c) $\frac{(1-\log x)}{(1+\log x)}$ (d) $\frac{(\log x)}{(1-\log x)^2}$

20. The mean and variance of a random variable X having a binomial distribution are 4 and 2 respectively, find the value of $P(X = 1)$.
(a) $\frac{1}{4}$ (b) $\frac{1}{16}$ (c) $\frac{1}{8}$ (d) $\frac{1}{32}$

21. Find the domain of the function $f(x) = \frac{(x^2+1)}{(x^2-3x+3)}$
(a) $R - \{1, 2\}$ (b) $R - \{1, 4\}$
(c) R (d) $R - \{1\}$

22. Let $A(2, -3)$ and $B(-2, 1)$ be vertices of a triangle ABC. If the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of the vertex C is the line
(a) $3x - 2y = 3$ (b) $2x + 3y = 9$
(c) $2x - 3y = 7$ (d) $3x + 2y = 5$

23. Find ${}^nC_{21}$ if ${}^nC_{10} = {}^nC_{11}$.
(a) 1 (b) 0 (c) 11 (d) 10

24. 9^{th} term in $\frac{n(n-4)}{(n^2+1)}$ is
(a) $\frac{55}{82}$ (b) $\frac{40}{82}$ (c) $\frac{45}{82}$ (d) $\frac{36}{82}$

25. Find area of the triangle with vertices (2, 3), (0, 1) and (1, 2).
(a) $\frac{1}{2}$ sq. unit (b) 1 sq. unit
(c) 2 sq. units (d) $2\frac{1}{2}$ sq. unit

26. Solve the system of linear equations $x + y = 1$, $x - 2y = 4$.
(a) $x = 2, y = -5$ (b) $x = 1, y = 2$
(c) $x = 2, y = -1$ (d) $x = -1, y = 2$

27. Find $\frac{dy}{dx}$ if $y = \sin^2 x + \cos^4 x$.

(a) $\frac{-\sin 4x}{4}$ (b) $\frac{-\sin 2x}{2}$

(c) $\frac{\sin 4x}{4}$ (d) $\frac{-\sin 4x}{2}$

28. A manufacturer has 500 L of a 10% solution of acid. Find the range of 40% acid to be added to it, such that the acid content in the resultant mixture will be more than 15% but less than 20% ?

(a) More than 250 L and less than 300

(b) More than 300 L and less than 400

(c) More than 100 L and less than 250

(d) More than 250 L and less than 350

29. If the eccentricity of an ellipse with its centre at the origin is 1/2 and one of its directrices is $x = 4$, then find the equation of the ellipse.

(a) $3x^2 + 4y^2 = 1$ (b) $4x^2 + 3y^2 = 12$

(c) $3x^2 + 4y^2 = 12$ (d) $4x^2 + 3y^2 = 12$

30. Integrate $\frac{\sec^2(\sin^{-1} x)}{\sqrt{1-x^2}}$

(a) $\sin(\tan^{-1} x) + c$ (b) $\tan(\sec^{-1} x) + c$

(c) $\tan(\sin^{-1} x) + c$ (d) $-\tan(\cos^{-1} x) + c$

31. A circle with centre (0, 3) is passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Find the radius of the circle.

(a) $\frac{7}{2}$ units (b) 3 units

(c) 4 units (d) $\sqrt{12}$ units

32. Which term of the series $\sqrt{2}, \frac{2}{3}, 2\frac{\sqrt{2}}{9}, \ldots$ is $\frac{16}{2187}$?

(a) 10^{th} term (b) 8^{th} term

(c) 9^{th} term (d) 11^{th} term

33. Integrate $\frac{1}{x^3(x^3-1)}$.

(a) $\frac{1}{3}\log\left|\frac{x^3}{x^3-1}\right| + C$ (b) $\frac{1}{3}\log\left|\frac{x^3-1}{x^3}\right| + C$

(c) $\log\left|\frac{x^3}{x^3-1}\right| + C$ (d) $\frac{1}{3}\log\left|\frac{x^3-1}{x^3}\right| + C$

34. Evaluate $\lim_{x\to 3} \frac{3-x}{\sqrt{4+x}-\sqrt{1+2x}}$

(a) 0 (b) $7\sqrt{2}$ (c) $4\sqrt{7}$ (d) $2\sqrt{7}$

35. $\int \sec^3 x\, dx =$

(a) $\frac{1}{2}[\sec x \tan x - \log|\sec x + \tan x|] + C$

(b) $\frac{1}{2}[\sec x \tan x + \log|\sec x - \tan x|] + C$

(c) $\frac{1}{2}[\sec x \tan x - \log|\sec x - \tan x|] + C$

(d) $\frac{1}{2}[\sec x \tan x + \log|\sec x + \tan x|] + C$

36. The sum of first three terms of a geometric progression is $\frac{7}{9}$ and their product is $-\frac{8}{27}$. Find the common ratio of the series.

(a) $r = -\frac{2}{3}$ or $-\frac{3}{2}$ (b) $r = -\frac{2}{3}$ or $\frac{3}{2}$

(c) $r = \frac{2}{3}$ or $-\frac{3}{2}$ (d) $r = \frac{2}{3}$ or $\frac{3}{2}$

37. Differentiate $\sin(\sin 2x)$.

(a) $2\cos 2x \cos 2x$ (b) $2\cos 2x \cos(\sin 2x)$

(c) $2\cos 2x \sin 2x$ (d) $\cos 2x \cos(\sin 2x)$

38. In a moderately asymmetrical distribution, the mean and median are 36 and 34 respectively, find out the value of empirical mode?

(a) 30 (b) 32 (c) 42 (d) 22

39. If $\sin y = x \sin(a + y)$, then find $\frac{dy}{dx}$.

(a) $\frac{\sin^2(a+y)}{\sin a}$ (b) $\frac{\sin a}{\sin^2(a+y)}$

(c) $\sin a \,.\, \sin^2(a-y)$ (d) $\frac{\sin^2(a-y)}{\sin a}$

40. Solve the inequality $2x - 5 \le \frac{(4x-7)}{3}$.

(a) $x \in (-\infty, 4)$ (b) $x \in (-\infty, 4]$

(c) $x \in (-\infty, 8]$ (d) $x \in (-\infty, -4]$

41. If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are three non-coplanar vectors then $(\vec{u}+\vec{v}-\vec{w})\cdot[(\vec{u}-\vec{v})\times(\vec{v}-\vec{w})]$ equals to

(a) $\vec{u}\cdot\vec{w}\times\vec{v}$ (b) $\vec{u}\cdot\vec{v}\times\vec{w}$

(c) 0 (d) $3\vec{u}\cdot\vec{v}\times\vec{w}$

42. The vector $\hat{i} + x\hat{j} + 3\hat{k}$ is rotated through an angle θ and doubled in magnitude, then it becomes $4\hat{i} + (4x-2)\hat{j} + 2\hat{k}$. Find the value of x.

(a) $\left\{-\frac{2}{3}, 2\right\}$ (b) $\left\{\frac{2}{3}, 2\right\}$

(c) $\left\{\frac{2}{3}, 0\right\}$ (d) $\left\{\frac{1}{3}, 2\right\}$

43. If $y = xe^{2y}$, then find $\frac{dy}{dx}$.

(a) $\frac{y}{x(1-2x)}$ (b) $\frac{x}{y(1-2x)}$

(c) $\frac{x}{y(1-2y)}$ (d) $\frac{y}{x(1-2y)}$

44. A line makes the same angle θ with each of the x and z axis. If it makes the angle β with y-axis such that $\sin^2\beta = 3\sin^2\theta$, then $\cos^2\theta$ equals

(a) $\frac{3}{5}$ (b) $\frac{1}{5}$ (c) $\frac{2}{5}$ (d) $\frac{2}{3}$

45. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$

(a) Touch each other

(b) Cut at an angle $\frac{\pi}{4}$

(c) Cut at an angle $\frac{\pi}{3}$

(d) Cut at an angle $\frac{\pi}{2}$

46. Solve for x, $\tan^{-1}\left(\frac{1}{x}\right) = \pi + \tan^{-1}x, 0 < x < 1$

(a) not defined (b) $\sqrt{3}$

(c) ± 1 (d) 0

47. Solve the equation $x^2 - \sqrt{3}x + 1 = 0$

(a) $x = \frac{(-\sqrt{3} \pm 2i)}{2}$

(b) $x = \frac{(-\sqrt{3} \pm i)}{2}$

(c) $x = (-\sqrt{3} \pm i)$

(d) $x = \frac{(\sqrt{3} \pm i)}{2}$

48. Differentiate $x^{-3}(2 + 7x)$.

(a) $\frac{-1}{x^4}(3+7x)$ (b) $\frac{-2}{x^3}(2+7x)$

(c) $\frac{2}{x^4}(2+7x)$ (d) $\frac{-2}{x^4}(3+7x)$

49. The probability that A speaks the truth is $\frac{3}{5}$ and probability that B speaks the truth is $\frac{3}{4}$. Find the probability that they contradict each other when asked to speak a fact.

(a) $\frac{3}{20}$ (b) $\frac{4}{5}$ (c) $\frac{9}{20}$ (d) $\frac{7}{20}$

50. Find the multiplicative inverse of $5 - 6i$.

(a) $-\left(\frac{5}{61}\right) + \left(\frac{6}{61}\right)i$ (b) $\left(\frac{5}{61}\right) + \left(\frac{6}{61}\right)i$

(c) $\left(\frac{5}{61}\right) - \left(\frac{6}{61}\right)i$ (d) $-\left(\frac{5}{61}\right) - \left(\frac{6}{61}\right)i$

51. Find the sum to n terms of the series whose n^{th} term is $n^2 - 2n$.

(a) $2n(n-1)$ (b) $2n(n+1)$

(c) $2n^2(n+1)$ (d) $2n^2(n-1)$

52. A plane which passes through the point (3, 2, 0) and the line $\frac{(x-4)}{1} = \frac{(y-7)}{5} = \frac{(z-4)}{4}$ is

(a) $x + y + z = 1$ (b) $x + 2y - z = 1$

(c) $x - y + z = 1$ (d) $2x - y + z = 5$

53. Find the value of $\sin(2\tan^{-1}x)$, $|x| \le 1$

(a) $\frac{1}{x}$ (b) x (c) $\frac{1}{x^2}$ (d) $\frac{2x}{1+x^2}$

54. In a group of students, 40 students choose Physics, 30 students choose Chemistry and 15 students choose both. Each of the students choose either Physics or Chemistry. How many students are there in the group?

(a) 60 (b) 70 (c) 75 (d) 55

55. Discuss the continuity of the function $f(x) = \sin 2x - 1$ at the point $x = 0$ and $x = \pi$.

(a) Continuous at $x = 0, \pi$

(b) Discontinuous at $x = 0$ but continuous at $x = \pi$

(c) Continuous at $x = 0$ but discontinuous at $x = \pi$

(d) Discontinuous at $x = 0, \pi$

56. The value of $2\cot^2\left(\frac{\pi}{6}\right)+4\tan^2\left(\frac{\pi}{6}\right)-3\operatorname{cosec}\frac{\pi}{6}$ is

(a) 2 (b) 4 (c) $\frac{4}{3}$ (d) $\frac{3}{4}$

57. If $A=\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}$, then $A^2+2A-3I=$

(a) $\begin{bmatrix}4 & -6\\ 6 & 4\end{bmatrix}$ (b) 0

(c) $\begin{bmatrix}-6 & 2\\ -2 & 6\end{bmatrix}$ (d) $5I$

58. The sum of first 8 terms of the geometric series $2+6+18+54+\ldots..$ is

(a) 6506 (b) 5650 (c) 6650 (d) 6560

59. Let $A=\{1, 2, 3, 4, 5\}$. Find the domain in the relation from A to A by $R=\{(x, y) : y=2x-1\}$.

(a) $\{1, 2, 3\}$ (b) $\{1, 2\}$
(c) $\{1, 3, 5\}$ (d) $\{2, 4\}$

60. Find $\cos\left(\frac{x}{2}\right)$, if $\tan x=\left(\frac{5}{12}\right)$, x is in Quadrant III.

(a) $\frac{5}{\sqrt{13}}$ (b) $\frac{5}{\sqrt{26}}$ (c) $\frac{5}{13}$ (d) $\frac{5}{26}$

61. $\int\sin^{-1}(\cos 3x)\,dx$ is

(a) $\frac{3\pi}{2}x-\frac{x^2}{2}+C$ (b) $\frac{\pi}{2}x-\frac{x^2}{2}+C$

(c) $\frac{3\pi}{2}x+\frac{x^2}{2}+C$ (d) $\frac{\pi}{2}x-\frac{x^3}{3}+C$

62. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{b}\times\vec{c}=\vec{a}$, then

(a) $\vec{a}\neq\vec{b}\neq\vec{c}$ (b) $\vec{a}=\vec{b}=\vec{c}$
(c) $\vec{a}\neq\vec{b}\neq\vec{c}\neq 1$
(d) $\vec{a}, \vec{b}, \vec{c}$ are orthogonal in pairs

63. Determine n if $2\,{}^nC_2 : {}^nC_2 = 9 : 2$

(a) 5 (b) 4 (c) 3 (d) 2

64. $\lim\limits_{x\to 0}\dfrac{\sin^{-1}(x-2)}{x^2-4}$ is equal to

(a) 0 (b) 2

(c) $\frac{1}{2}$ (d) 1

65. If $a\neq 0$ and the line $2bx+3cy+4d=0$ passes through the points of intersection of the parabolas $y^2=4ax$ and $x^2=4ay$, then

(a) $d^2+(2b-3c)^2=0$
(b) $d^2+(3b-2c)^2=0$
(c) $d^2+(2b+3c)^2=0$
(d) $d^2+(3b+2c)^2=0$

66. If $f(x)=x^3-1$, then find $\dfrac{[f(3)+f(2)]}{11}$.

(a) 1 (b) 2 (c) 3 (d) 4

67. The intercept on the line $y=x$ by the circle $x^2+y^2-2x=0$ is AB. Find the equation of the circle on AB as a diameter.

(a) $x^2+y^2+x+y=0$ (b) $x^2+y^2-x-y=0$
(c) $x^2+y^2+x-y=0$ (d) $x^2+y^2-x+y=0$

68. Compute the product $\begin{bmatrix}-1 & 3 & 0\\ 2 & 1 & 4\end{bmatrix}\times\begin{bmatrix}1 & 3\\ 0 & 2\\ -2 & -1\end{bmatrix}$

(a) $\begin{bmatrix}1 & 3\\ 6 & -4\end{bmatrix}$ (b) $\begin{bmatrix}-1 & -3\\ 6 & 4\end{bmatrix}$

(c) $\begin{bmatrix}-1 & 3\\ -6 & 4\end{bmatrix}$ (d) $\begin{bmatrix}-1 & 3\\ -6 & -4\end{bmatrix}$

69. Find the values of x, y and z from the following equations : $\begin{bmatrix}4 & x-z\\ 2+y & xz\end{bmatrix}=\begin{bmatrix}4 & 3\\ -1 & 10\end{bmatrix}$

(a) $x=-5, y=3, z=2$
(b) $x=5, y=-3, z=2$
(c) $x=5, y=3, z=-2$
(d) $x=5, y=-3, z=-2$

70. Let $A=\{1, 2, \{a, b\}, 3, 4\}$ which among the following statements is incorrect?

(a) $\{a, b\}\subset A$ (b) $\{a, b\}\in A$
(c) $\{\{a, b\}\}\subset A$ (d) $\phi\in A$

71. The probability of India winning a test match against South Africa is $\frac{1}{2}$ assuming independence from match to match played. The probability that in a match series India's second win occurs at the third one day is

(a) $\frac{1}{8}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{2}{3}$

72. In an experiment with 15 observations on x the following results are available $\Sigma x^2 = 2830$, $\Sigma x = 170$. One observation that was 20, was found to be wrong and was replaced by correct value 30. Find the correct variance.
(a) 78 (b) 186 (c) 158 (d) 18

73. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is
(a) $2ax + 2by - (a^2 + b^2 + 4) = 0$
(b) $2ax + 2by + (a^2 + b^2 + 4) = 0$
(c) $2ax - 2by + (a^2 + b^2 + 4) = 0$
(d) $2ax - 2by - (a^2 + b^2 + 4) = 0$

74. Find the radius of the circle in which the sphere $x^2 + y^2 + z^2 + 2x - 2y - 4z = 19$ is cut by the plane $x + 2y + 2z + 7 = 0$
(a) 2 (b) 3 (c) 1 (d) 4

75. Let $A = \{1, 2, 3, 4\}$ and R be the relation on A defined by $\{(a, b) : a, b \in A, a \times b$ is an even number$\}$, then find the range of R.
(a) $\{1, 2, 3, 4\}$ (b) $\{2, 4\}$
(c) $\{2, 3, 4\}$ (d) $\{1, 2, 4\}$

ANSWER KEYS

1. (d)	**2.** (d)	**3.** (a)	**4.** (c)
5. None	**6.** (d)	**7.** (d)	**8.** (a)
9. (c)	**10.** (d)	**11.** (b)	
12. The question is wrong.			
13. (b)	**14.** (b)	**15.** (c)	**16.** (c)
17. (c)	**18.** (d)	**19.** (a)	**20.** (d)
21. (c)	**22.** (b)	**23.** (a)	**24.** (c)
25. None	**26.** (c)	**27.** (d)	**28.** (c)
29. (c)	**30.** (c)	**31.** (c)	**32.** (b)
33. None	**34.** (d)	**35.** (d)	**36.** (a)
37. (b)	**38.** (a)	**39.** (a)	**40.** (b)
41. (b)	**42.** (a)	**43.** (d)	**44.** (a)
45. (d)	**46.** (c)	**47.** (d)	**48.** (d)
49. (c)	**50.** (b)	**51.** None	**52.** (c)
53. (d)	**54.** (d)	**55.** (a)	**56.** (c)
57. (a)	**58.** (d)	**59.** (a)	**60.** None
61. None	**62.** (d)		
63. The question is wrong.			
64. None	**65.** (c)	**66.** (c)	**67.** (b)
68. (c)	**69.** (b)	**70.** (a, d)	**71.** (c)
72. (a)	**73.** (a)	**74.** (b)	**75.** (a)

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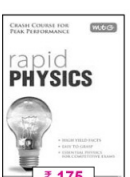

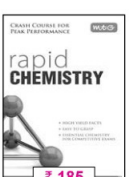

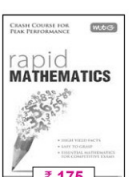

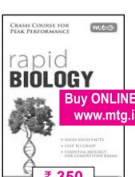

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Q1. Let f be a twice differentiable function such that $f''(x) = -f(x)$, $f'(x) = g(x)$ and $h(x) = [f(x)]^2 + [g(x)]^2$. Find $h(10)$, if $h(5) = 11$.

[Devika, Kochi]

Ans. Given, $h(x) = [f(x)]^2 + [g(x)]^2$

$\Rightarrow \quad h'(x) = 2f(x)\cdot f'(x) + 2g(x)\cdot g'(x)$

$= 2[f(x)\cdot g(x) - g(x)\cdot f(x)]$

$= 0$

$[\because f'(x) = g(x) \text{ and } g'(x) = f''(x) = -f(x)]$

$\therefore \quad h'(x) = 0 \; \forall \; x$

$\therefore$ $h(x)$ is constant. $\Rightarrow \quad h(10) = h(5) = 11.$

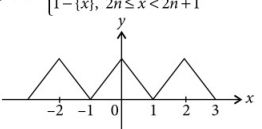
Q2. For any real number x, let $|x|$ denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by

$$f(x) = \begin{cases} x - [x] & , \text{ if } [x] \text{ is odd} \\ 1 + [x] - x \, , & \text{if } [x] \text{ is even} \end{cases}$$

Then find the value of $\dfrac{\pi^2}{10}\int\limits_{-10}^{10} f(x)\cos \pi x\, dx.$

[Rajkumar, Patna]

Ans. $f(x) = \begin{cases} \{x\}, & 2n-1 \le x < 2n \\ 1-\{x\}, & 2n \le x < 2n+1 \end{cases}$



We have $f(x)$ is periodic

$$I = \int\limits_{-10}^{10} f(x)\cos \pi x\, dx = 2\int\limits_{0}^{10} f(x)\cos \pi x\, dx$$

$$I = 2\times 5\int\limits_{0}^{2} f(x)\cos \pi x\, dx$$

$$I = 10\left[\int\limits_{0}^{1}(1-x)\cos \pi x\, dx + \int\limits_{1}^{2}(x-1)\cos \pi x\, dx\right]$$

$$= 10\,[(I_1) + (I_2)]$$

$$I_2 = \int\limits_{1}^{2}(x-1)\cos \pi x\, dx$$

$$I_2 = -\int\limits_{0}^{1} t\cos \pi t\, dt \quad (x-1=t)$$

We have $I_1 = \int\limits_{0}^{1}(1-x)\cos \pi x\, dx = -\int\limits_{0}^{1} x\cos \pi x\, dx$

$$I = 10\left(-2\int\limits_{0}^{1} x\cos \pi x\, dx\right)$$

$$= -20\left[\frac{x\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_0^1$$

$$= -20\left[-\frac{1}{\pi^2} - \frac{1}{\pi^2}\right] = \frac{40}{\pi^2}$$

Hence, $\dfrac{\pi^2}{10}\int\limits_{-10}^{10} f(x)\cos \pi x\, dx = \dfrac{\pi^2}{10}\times\dfrac{40}{\pi^2} = 4$

Q3. Find the sum of the series

$$\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \ldots \text{upto } m \text{ terms}\right]$$

[Latha, W.B.]

Ans. $\displaystyle\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \ldots \text{ upto } m \text{ terms}\right]$

$$= \sum_{r=0}^{n} (-1)^r \, {}^nC_r \left(\frac{1}{2}\right)^r + \sum_{r=0}^{n} (-1)^r \, {}^nC_r \left(\frac{3}{4}\right)^r +$$

$$\sum_{r=0}^{n} (-1)^r \, {}^nC_r \left(\frac{7}{8}\right)^r + \ldots \text{ upto } m \text{ terms}$$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \ldots \text{ upto } m \text{ terms}$$

$$\left\{\text{using } \sum_{r=0}^{n} (-1)^r \, {}^nC_r x^r = (1-x)^n\right\}$$

$$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n + \ldots \text{ upto } m \text{ terms}$$

$$= \frac{1}{2^n}\left(1 + \frac{1}{2^n} + \frac{1}{4^n} + \ldots \text{ upto } m \text{ terms}\right)$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{1-\left(\frac{1}{2^n}\right)^m}{1-\frac{1}{2^n}}\right] = \frac{2^{mn}-1}{2^{mn}(2^n-1)}.$$

Q4. Let $A_n = \left(\frac{3}{4}\right) - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$

$B_n = 1 - A_n$. Find a least odd natural number n_0, so that $B_n > A_n$, $\forall\ n \geq n_0$.

[Rahil, U.P.]

Ans. $B_n = 1 - A_n > A_n \Rightarrow A_n < \frac{1}{2}$

$$\text{Now, } A_n = \frac{\frac{3}{4}\left(1-\left(-\frac{3}{4}\right)^n\right)}{1+\frac{3}{4}} < \frac{1}{2}$$

$$\Rightarrow \left(-\frac{3}{4}\right)^n > -\frac{1}{6}$$

Obviously, it is true for all even values of n

But for $n = 1$, $-\frac{3}{4} < -\frac{1}{6}$

$n = 3$, $\left(-\frac{3}{4}\right)^3 = -\frac{27}{64} < -\frac{1}{6}$

$n = 5$, $\left(-\frac{3}{4}\right)^5 = -\frac{243}{1024} < -\frac{1}{6}$

for $n = 7$, $\left(\frac{-3}{4}\right)^7 = \frac{-2187}{16384} > \frac{-1}{6}$

Hence, minimum odd natural number $n_0 = 7$.

Q5. Integrate $\sin x \cdot \sin 2x \cdot \sin 3x + \sec^2 x \cdot \cos^2 2x + \sin^4 x \cdot \cos^4 x$.

[Vikas, M.P.]

Ans. Let $I_1 = \int \sin x \sin 2x \sin 3x \, dx$

$$= \frac{1}{4}\int (\sin 4x + \sin 2x - \sin 6x)\, dx$$

$$= -\frac{\cos 4x}{16} - \frac{\cos 2x}{8} + \frac{\cos 6x}{24}$$

$$I_2 = \int \sec^2 x \cdot \cos^2 2x \, dx$$

$$= \int \sec^2 x (2\cos^2 x - 1)^2 \, dx$$

$$= \int (4\cos^2 x + \sec^2 x - 4)\, dx$$

$$= \int (2\cos 2x + \sec^2 x - 2)\, dx$$

$$= \sin 2x + \tan x - 2x$$

and $I_3 = \int \sin^4 x \cos^4 x \, dx$

$$= \frac{1}{128}\int (3 - 4\cos 4x + \cos 8x)\, dx$$

$$= \frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024}$$

$$\therefore \quad I = I_1 + I_2 + I_3$$

$$= -\frac{\cos 4x}{16} - \frac{\cos 2x}{8} + \frac{\cos 6x}{24}$$

$$+ \sin 2x + \tan x - 2x + \frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024}.$$

MATHS MUSING

SOLUTION SET-140

1. (d) : The lines are $\frac{x}{1}=\frac{y}{1}=\frac{z-1}{0}, \frac{x}{1}=\frac{y}{-1}=\frac{z+1}{0}$.

If $c(\alpha, \beta, \gamma)$, the distance of c from the lines are

$\sqrt{(\gamma-1)^2+\frac{(\alpha-\beta)^2}{2}}$ and $\sqrt{(\gamma+1)^2+\frac{(\alpha+\beta)^2}{2}}$

$\Rightarrow \alpha\beta + 2\gamma = 0$ or $xy + 2z = 0$.

2. (c): $\binom{19}{3}+\binom{18}{3}+....+\binom{3}{3}=$

coeff. of x^3 in $(1+x)^3 + (1+x)^4 + \ldots + (1+x)^{19}$

$=$ coeff. of x^4 in $(1+x)^3((1+x)^{17}-1) = \binom{20}{4}$

$= 3 \cdot 5 \cdot 17 \cdot 19$

3. (a): There are 9 sets of 6 terms giving

$9(7 + 19 + 31 + 43 + 55 + 67 + 79 + 91 + 103)$

$= 3^4 \cdot 5 \cdot 11$

4. (d) : $P(x) = ax^3 + bx^2 + cx + d$

$P''(1) = 0, P'(-1) = 0$

$\Rightarrow P'(x) = 3a(x+1)(x-3)$

$\therefore$ $P(x)$ has minimum at $x = 3$.

5. (c): Let $a + ib = re^{i\theta}$

$\Rightarrow (re^{i\theta})^{c+id} = p$

$(c + id)(\ln r + i\theta) = \ln p$

$c\cdot\ln r - d\theta = \ln p, d\cdot\ln r = -c\theta$

Eliminating θ, $(c^2 + d^2)\ln r = c \cdot \ln p$

$\Rightarrow (a^2 + b^2)^{(c^2+d^2)} = p^{2c}$

6. (a),(b): We apply mean value theorem to $f[x] = x^{1/5}$ in $[5^5, 5^5 + 1]$

$(5^5 + 1)^{1/5} - 5 = \frac{1}{5c^{4/5}}$. But $c \in (5^5, 5^5 + 1)$

$\therefore \quad 5^5 - 5c^{4/5} < 5(5^5 + 1)^{4/5} < 5^5 + 1$

$5^5 < ((5^5 + 1)^{1/5} - 5)^{-1} < 5^5 + 1 = 3126 = 2\cdot3\cdot521$

7. (c): $A(t_1), B(t_2), C(t_3)$ are the vertices, $S(a, 0)$ in the focus.

$SA \perp BC$, $SB \perp CA$

$\Rightarrow \Sigma t_1t_2 = -5, \Sigma t_1 = -t_1t_2t_3$

$G(x, y)$ is the centroid

$\Rightarrow \frac{3x}{a} = \Sigma t_1{}^2, \frac{3y}{2a} = \Sigma t_1,$

$\Rightarrow y^2 = \frac{4a}{3}\left(x - \frac{10a}{3}\right)$, since $\Sigma t_1t_2 = -5$

Latusrectum is $\frac{4a}{3}$.

8. (a):

S divides OG in the ratio $3 : -1$

$\Rightarrow$ The locus of the circumcentre is

$y^2 = 2a\left(x - \frac{9a}{2}\right)$ with latusrectum $2a$.

9. (8): D, X, Y lie on incircle.

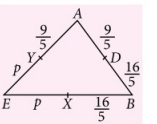
In triangle ABE

$s = p + 5$

$r = \frac{6}{5}$

$\Delta^2 = (p+5)p \cdot \frac{144}{25}$

$\Rightarrow p = \frac{5}{3}, \Delta = 8$



10. (c): $(1 + x + x^2)^n = (1 + x(1 + x))^n$

$= 1 + \binom{n}{1} x(1 + x) + \binom{n}{2} x^2 (1 + x)^2 + \binom{n}{3} x^3 (1 + x)^3 + \ldots$

P. $n = 3, 1 + 3x(1 + x) + 3x^2(1 + x)^2 + x^3(1 + x)^3$

coeff. $x^3 = 6 + 1 = 7$

Q. $n = 4, 1 + 4x(1 + x) + 6x^2(1 + x)^2 + 4x^3(1 + x)^3 + x^4(1 + x)^4$

coeff. $x^4 = 6 + 12 + 1 = 19$

R. $n = 5, 1 + 5x(1 + x) + 10x^2(1 + x)^2 + 10x^3(1 + x)^3 + 5x^4(1 + x)^4 + x^5(1 + x)^5$

coeff. $x^5 = 30 + 20 + 1 = 51$

S. $n = 6, 1 + 6x(1 + x) + 15x^2(1 + x)^2 + 20x^3(1 + x)^3 + 15x^4(1 + x)^4 + 6x^5(1 + x)^5 + x^6(1 + x)^6$

coeff. $x^6 = 20 + 90 + 30 + 1 = 141$

Solution Sender of Maths Musing

SET-139

1. Gouri Sankar Adhikari, Mayta, Paschim Medinipur
2. Chandan Pramanik, Mayta, Paschim Medinipur

SET-140

1. Devjit Acharjee, Kolkata, W.B.
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MOCK TEST PAPER 2015

JEE Main & Advanced

Series-3

Permutations and Combinations : Fundamental principle of counting, permutation as an arrangement and combination as selection, meaning of $p(n, r)$ and $c(n, r)$, simple applications.

Logarithms : Logarithms and their properties.

Co-ordinate geometry-2D : Equation of a circle in various forms, equations of tangent, normal and chord. Parametric equations of a circle, intersection of a circle with a straight line or a circle, equation of a circle through the point of intersection of two circles and those of a circle and a straight line.

PAPER-I

SECTION-I

Single Option Correct

This section contains 10 multiple choice questions. Each question has 4 choices, out of which ONLY ONE is correct.

1. If $\frac{{}^{n}P_{r-1}}{a} = \frac{{}^{n}P_{r}}{b} = \frac{{}^{n}P_{r+1}}{c}$, then

(a) $b^2 = a(b + c)$ (b) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$

(c) $a^2 = c(a + b)$ (d) $abc = 1$

2. If $\frac{1}{2} \le \log_{0.1} x \le 2$, the maximum value of x is

(a) $\frac{1}{10}$ (b) $\frac{1}{\sqrt{10}}$ (c) $\frac{1}{100}$ (d) 1

3. m points on one straight line are joined to n points on another straight line. The number of intersection of the line segments thus so formed is

(a) ${}^{m}C_2 \div {}^{n}C_2$ (b) $\frac{mn(m-1)(n-1)}{4}$

(c) $\frac{{}^{m}C_2 \cdot {}^{n}C_2}{2}$ (d) ${}^{m}C_2 + {}^{n}C_2$

4. If $a_n > 1 \; \forall \; n \in N$, then

$\log_{a_2} a_1 + \log_{a_3} a_2 + ... + \log_{a_n} a_{n-1} + \log_{a_1} a_n$

has the minimum value

(a) 1 (b) 2

(c) 0 (d) none of these

5. Two circles have equations $x^2 + y^2 - 4x - 6y - 8 = 0$ and $x^2 + y^2 - 2x - 3 = 0$, then

(a) they cut each other

(b) they touch each other

(d) one circle lies inside the other

(d) one circle lies wholly outside the other

6. A variable circle C has the equation $x^2 + y^2 - 2(t^2 - 3t + 1)x - 2(t^2 + 2t)y + t = 0$, where t is a parameter. The locus of the centre of the circle is/an

(a) parabola (b) ellipse

(c) hyperbola (d) pair of straight lines

7. The circle $x^2 + y^2 + 4x + 8y + 5 = 0$ intersects the line $3x - 4y = m$ at two distinct points if

(a) $-35 < m < 15$ (b) $15 < m < 65$

(c) $35 < m < 85$ (d) $-85 < m < -35$

8. A class contains three girls and four boys. Every Saturday five students go on a picnic, a different group being sent each week. During the picnic, each girl in the group is given a doll by the accompanying teacher. After all possible groups of the five have gone once, the total no. of dolls the girls have got is

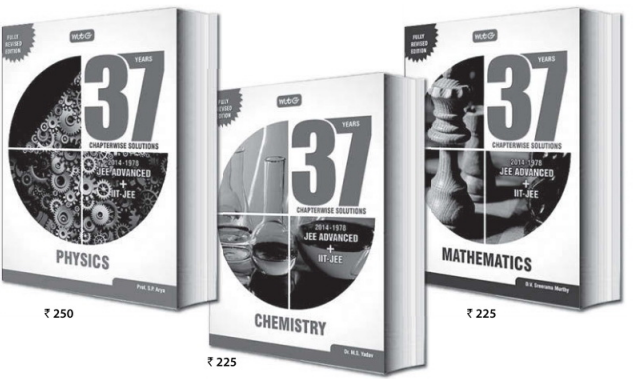
(a) 27 (b) 11 (c) 21 (d) 45

9. The number of zeros at the end of 127! is

(a) 28 (b) 29 (c) 30 (d) 31

By : Sankar Ghosh, HOD(Math), Takshyashila. Mob : 09831244397

Mad about rehearsing?



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10. An isosceles triangle with base 24 and legs 15 each is inscribed in a circle with centre (–1, 1), then locus of the centroid of the triangle is

(a) $4(x^2 + y^2) + 8x - 8y - 73 = 0$

(b) $2(x^2 + y^2) + 4x - 4y - 31 = 0$

(c) $2(x^2 + y^2) + 4x - 4y - 21 = 0$

(d) $4(x^2 + y^2) + 8x - 8y - 161 = 0$

SECTION-II

Multiple Correct Options

This section contains 5 multiple choice questions. Each question has 4 choices, out of which ONE or MORE is/are correct.

11. If $\log_2|4 - 5x| < 2$, then $x \in$

(a) $\left(0, \frac{5}{4}\right)$ (b) $\left(0, \frac{4}{5}\right)$

(c) $\left(0, -\frac{5}{4}\right)$ (d) $\left(\frac{4}{5}, \frac{8}{5}\right)$

12. If $10! = 2^p \cdot 3^q \cdot 5^r \cdot 7^s$, then

(a) $2q = p$ (b) $pqrs = 64$

(c) 10! is divisible by 280

(d) the product of two integers in 10! is 135, when 10! is arranged in a row

13. A circle passes through the points (–1, 1), (0, 6) and (5, 5). The point(s) on this circle, the tangent(s) at which is/are parallel to the straight line joining the origin to its centre is/are

(a) (1, –5) (b) (5, 1)

(c) (–5, –1) (d) (–1, 5)

14. A ray of light passing through (–3, 1) reflects on x-axis at A and reflected ray is tangential to the circle $x^2 + y^2 = 1$ at P.

(a) The reflected ray is $3x - 4y + 5 = 0$

(b) $A\left(\frac{5}{3}, 0\right)$ (c) $P\left(-\frac{3}{5}, 0\right)$

(d) Area of the ΔAPO is $\frac{3}{2}$ square units ('O' is origin).

15. Two class rooms A and B have capacity of 25 and $n - 25$ seats respectively. A_n denotes the number of possible seating arrangements of room 'A', when 'n' students are to be seated in these rooms, starting from room 'A' which is to be filled up to its capacity. If $A_n - A_{(n-1)} = 25!(^{49}C_{25})$, then '$n$' equals

(a) 50 (b) 48 (c) 49 (d) 51

SECTION-III

Assertion & Reason Type

This section contains 5 questions. Each question contains Statement-1 (Assertion) and Statement-2 (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

(a) If both A and R is correct and R is the proper reason of A.

(b) If both A and R is correct and R is not the proper reason of A.

(c) If A is correct and R is wrong.

(d) If A is wrong and R is correct.

16. **Statement-1 :** The equation $(x + 2)^2 + (y - 3)^2 = -1$ does not represent any circle.
Statement-2 : The equation $(x + 2)^2 + (y - 3)^2 = -1$ does not represent any real locus.

17. **Statement-1 :** Let N is the number of 3-digit numbers with distinct digits so that the digits in any number neither increase nor decrease. Then sum of the divisor of N is 1064.
Statement-2 : If p_1, p_2, p_3 are primes then the sum of the divisors of $N = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3}$ is $\frac{(p_1^{\alpha_1+1} - 1)(p_2^{\alpha_2+1} - 1)(p_3^{\alpha_3+1} - 1)}{(p_1 - 1)(p_2 - 1)(p_3 - 1)}$.

18. In a shop there are five types of ice-creams available. A child buys six ice-creams.
Statement-1 : The number of different ways the child can buy the six ice-creams is $^{10}C_5$.
Statement-2 : The number of different ways the child can buy six ice-creams is equal to the number of different ways of arranging 6 A's and 4B's in a row.

19. $ABCD$ is a quadrilateral in which a circle is inscribed.
Statement-1 : The lengths of the sides of the quadrilateral are in A.P.
Statement-2 : The lengths of tangents from an external point to a circle are equal.

20. Let a, b, c are respectively the p^{th}, q^{th} and r^{th} terms of G.P.
Statement-1 : $(q - r)\log a + (r - p)\log b + (q - r)\log c = 0$
Statement-2 : If the first term and common ratio of a G.P be respectively α and β, then its general term is defined as $T_n = \alpha\beta^{(n-1)}$.

SECTION-IV

Integer Answer type

This section contains 5 questions. The answer to each of the questions is a single-digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2, respectively, then the correct darkening of bubbles will look like the following.

21. The value of $\left[\frac{1}{6}\left(\frac{3\log 1728}{1+\frac{1}{2}\log 0.36+\frac{1}{3}\log 8}\right)^{1/2}\right]^{-1}$ is

22. The solution of the equation $\log_7 \log_5(\sqrt{x+5}+\sqrt{x}) = 0$ is

23. A circle touches x-axis at $(2, 0)$ and has intercept of 4 units on y-axis. The equations of the required circles are $x^2 + y^2 - 4x \pm k\sqrt{2} + 4 = 0$. The value of k is

24. The sum of the abscissa and ordinate of a point on the circle $x^2 + y^2 - 4x + 2y - 20 = 0$, which is nearest to $\left(2, \frac{3}{2}\right)$ is

25. The unit place of the number of numbers less than 1000 and divisible by 5 which can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 each digit not occurring more than once in each number, is

PAPER-II

SECTION-I

Single Option Correct

This section contains 10 multiple choice questions numbered 1 to 4. Each question has 4 choices (a), (b), (c)and (d), out of which ONLY ONE is correct.

1. Two gas companies G_1 and G_2, situated at (40, 0) and (0, 30) respectively (unit of distance = 1 km), offer to install equally priced gas ovens at a buyer's house. The company G_1 adds a charge of ₹ 40/km of aerial distance between the company and the buyer's house while it is ₹ 60 incase of the company G_2. Then the region where it is cheaper to install the oven from the gas company G_1 is

(a) The interior of the circle $(x + 32)^2 + (y - 54)^2 = 3600$

(b) The interior of the circle $(x - 54)^2 + (y + 32)^2 = 3600$

(c) The exterior of the circle $(x + 32)^2 + (y - 54)^2 = 3600$

(d) The exterior of the circle $(x - 54)^2 + (y + 32)^2 = 3600$

2. ${}^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r$ is equal to

(a) ${}^{n+2}C_{r+1}$ (b) ${}^{n+1}C_r$

(c) ${}^{n+1}C_{r+1}$ (d) ${}^{n+2}C_r$

3. If the two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points then

(a) $r < 2$ (b) $r = 2$ (c) $r > 2$ (d) $2 < r < 8$

4. The number of ways in which 9 flowers, of which 5 are identical and white, and the other four are of different colors, can be set on a garland so that no two of the colored flowers are consecutive, is

(a) 60 (b) 120

(c) 180 (d) none of these

5. A square is inscribed in the circle $x^2 + y^2 - 2x - 4y = 0$ whose sides are parallel to the axes of reference. A vertex of the square is

(a) (3, 1)

(b) (−2, 3)

(c) $\left(\frac{\sqrt{2}-\sqrt{5}}{\sqrt{2}}, \frac{2\sqrt{2}-\sqrt{5}}{\sqrt{2}}\right)$

(d) $\left(\frac{\sqrt{5}+\sqrt{2}}{\sqrt{2}}, -\frac{2\sqrt{2}+\sqrt{5}}{\sqrt{2}}\right)$

6. The least number of sides of a polygon in which the number of diagonals is at least 10 more than the number of sides, is

(a) 8 (b) 10 (c) 11 (d) 25

7. If the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 2\lambda y = 4$ have only one common tangent then λ is
(a) 1 (b) -1 (c) 0 (d) 2

8. The number of ways in which the five digits 1, 2, 3, 4, 5 can be arranged to make a number greater than 10000 such that the odd digits are in the ascending order, is
(a) 10 (b) 20 (c) 120 (d) 60

9. The sum of the solutions of the equation $|x-1|^{\log_2 x^2 - 2\log_x 4} = (x-1)^7$ is
(a) 10 (b) 18 (c) 20 (d) 30

10. The number of solutions of the equation $\log_4(x-1) = \log_2(x-3)$ is
(a) 3 (b) 1 (c) 2 (d) 0

SECTION-II

Multiple Correct Type

This section contains 5 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONE OR MORE is/are correct.

11. The equation $x^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$ has
(a) at least one negative solution
(b) exactly one irrational solution
(c) exactly three real solutions
(d) two non-real complex roots

12. The solution of the equation
$$\log_{10} x - \log_{10}\sqrt{x} = \frac{2}{\log_{10} x} \text{ is}$$
(a) 100 (b) 200 (c) $\frac{1}{100}$ (d) $\frac{1}{200}$

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13. If n number of straight lines satisfy the equation
$${}^{(n+5)}P_{(n+1)} = \frac{11(n-1)}{2} \cdot {}^{(n+3)}P_n,$$ then the number of points of intersections of the straight lines will be
(a) 15 (b) 28 (c) 21 (d) 10

14. Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal then which of the following equations can represent L_1?
(a) $x + y = 0$ (b) $x - y = 0$
(c) $x + 7y = 0$ (d) $x - 7y = 0$

15. The parametric equation of a circle is given by $x = 3\cos\theta + 2$, $y = 3\sin\theta$. Then its
(a) centre $= (-2, 0)$ (b) radius $= 3$
(c) centre $= (2, 0)$ (d) radius $= 1$

SECTION-III

Linked Comprehension Type

This section contains 2 paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE OR MORE is/are correct.

$P_{16\text{-}18}$: The radius of the circumscribed circle of the rectangle $ABCD$ is $x - 4y + 7 = 0$ and two vertices of the rectangle are respectively $(-3, 4)$, $(5, 4)$.

16. The centre of the circle is
(a) (1, 1) (b) (2, 2)
(c) (1, 2) (d) (2, 1)

17. The equation of the circle is
(a) $x^2 + y^2 - 2x - 4y - 15 = 0$
(b) $x^2 + y^2 + 2x - 4y - 16 = 0$
(c) $x^2 + y^2 + 2x + 4y - 4 = 0$
(d) none of these

18. If the point $(a, 0)$ lies inside the circumscribed circle then one possible value of a is
(a) 0 (b) 2
(c) 4 (d) infinitely many

$P_{19\text{-}21}$: There are 5 boys and 5 girls. The number of ways they can be seated in a row such that

19. These girls are together is
(a) 86400 (b) 28800
(c) 2880 (d) 7260

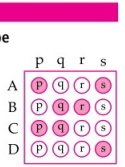
20. The boys and girls in alternate seats is
(a) 86400 (b) 28800
(c) 2880 (d) 7260

21. The number of ways they can form a circle such that the boys and girls sit in alternate seats, is
(a) 86400 (b) 28800
(c) 2880 (d) 7260

SECTION-IV

Matrix-Match Type

This section contains 2 questions. Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r and s. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example. If the correct matches are A-p and s ; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the above.



22. Match the following:
Consider all possible permutations of the letters of the word ENDEANOEL

Column I		Column II	
(A)	The number of permutations of the word ENDEA is	(p)	5!
(B)	The number of permutations in which the letter E occurs in the first and the last position is	(q)	$2 \times 5!$
(C)	The number of permutations in which none of the letters D,L,N occurs in the last five positions is	(r)	$7 \times 5!$
(D)	The number of permutations in which the letters A,E,O occurs only in odd positions is	(s)	$21 \times 5!$

23. Match the following:

Column I		Column II	
(A)	The solution set of $\lvert x-1\rvert^{\log_2(4-x)} < \lvert x-1\rvert^{\log_2(1+x)}$ is	(p)	$(-3, 1) \cup (3, 4)$
(B)	The value of $\log_{20}3$ lies in	(q)	$\left(-2,-\frac{3}{2}\right)\cup\left(\frac{8}{3},\infty\right)$
(C)	The solution set of the inequality $\log_{\left(x+\frac{5}{3}\right)}\left(\frac{x-5}{2x-3}\right)^2 < 0$ is	(r)	$\left(0,\frac{3}{2}\right) \cup (2, 4)$
(D)	If $a=\frac{25-x^2}{16}$ and $\log_a \frac{(24-2x-x^2)}{14} > 1$, then $x \in$	(s)	$\left(\frac{1}{2},\frac{1}{3}\right)$

ANSWERS

PAPER-1

1. (a) **2.** (b) **3.** (b) **4.** (d) **5.** (a)
6. (a) **7.** (a) **8.** (d) **9.** (d) **10.** (d)
11. (b, d) **12.** (a, b, c, d) **13.** (b, d)
14. (a, c, d) **15.** (a) **16.** (d) **17.** (a)
18. (d) **19.** (a) **20.** (a) **21.** (2) **22.** (4)
23. (4) **24.** (6) **25.** (4)

PAPER-2

1. (c) **2.** (a) **3.** (d) **4.** (d) **5.** (d)
6. (a) **7.** (c) **8.** (b) **9.** (b) **10.** (b)
11. (b, c) **12.** (a, c) **13.** (a, c) **14.** (b, d)
15. (b, c) **16.** (c) **17.** (a) **18.** (b) **19.** (a)
20. (b) **21.** (c)
22. A - p, B - s, C - q, D - q
23. A - r, B - s, C - q, D - p

ENRICH YOUR CONCEPTS ON XI

Complex Numbers and Quadratic Equations & Linear Inequalities

Based on NCERT Pattern

Series-4

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

DEFINITION

Any number of the form $z = a + ib$, where a, b are real numbers and $i = \sqrt{-1}$, is said to be a complex number. If $z = a + ib$ is a complex number, then 'a' is called the real part, written as Re(z) and 'b' is called the imaginary part, written as Im(z).

EQUALITY OF TWO COMPLEX NUMBERS

Any two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + i\,b_2$ are said to be equal if $a_1 = a_2$ and $b_1 = b_2$ *i.e.* Re (z_1) = Re (z_2) and Im (z_1) = Im (z_2).

ALGEBRA OF COMPLEX NUMBERS

1. Addition of two Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers. Then the sum $z_1 + z_2 = (a + c) + i(b + d)$

Name	Properties
Closure law	$z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2
Commutative law	$z_1 + z_2 = z_2 + z_1$, where z_1 and z_2 are complex numbers.
Associative law	$(z_1 + z_2) + z_3 = z_1 + i\,(z_2 + z_3)$, where z_1, z_2 and z_3 are complex numbers.
Existence of additive identity	There exists a complex number $0 + i0$ where 0 is called additive identity.
Existence of additive inverse	To every complex number $z = a + ib$, we have $-a + i(-b)$ (denoted as $-z$) called additive inverse or negative of z.

2. Difference of two Complex Numbers

Given any two complex numbers z_1 and z_2, then $z_1 - z_2 = z_1 + (-z_2)$

3. Multiplication of two Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then the product z_1z_2 is defined as $z_1z_2 = (ac - bd) + i(ad + bc)$

Name	Properties
Closure law	The product of two complex numbers is a complex number, where z_1 and z_2 are complex numbers.
Commutative law	For any two complex numbers z_1 and z_2, $z_1z_2 = z_2z_1$
Associative law	For any three complex numbers z_1, z_2, z_3, $(z_1z_2)z_3 = z_1(z_2z_3)$
Existence of multiplicative identity	There exists the complex number $1 + i0$ (denoted as 1), called the multiplicative identity such that $z \cdot 1 = z$, for every complex number z.
Existence of multiplicative inverse	For every non-zero complex number $z = a + ib$ or $a + bi (a \neq 0, b \neq 0)$, we have the complex number $\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$ (denoted by $\frac{1}{z}$ or z^{-1}), called the multiplicative inverse of z such that $z \cdot \frac{1}{z} = 1$ (the multiplicative identity)
Distributive law	For any three complex numbers z_1, z_2, z_3, we have (a) $z_1\,(z_2 + z_3) = z_1z_2 + z_1z_3$ (b) $(z_1 + z_2)\,z_3 = z_1z_3 + z_2z_3$

4. Division of two Complex Numbers

Given any two complex numbers z_1 and z_2, where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined by $\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$.

IDENTITIES OF COMPLEX NUMBERS

If z_1 and z_2 are two complex numbers, then

(i) $(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2$

(ii) $(z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$

(iii) $(z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$

(iv) $(z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$

(v) $(z_1^2 - z_2^2) = (z_1 + z_2)(z_1 - z_2)$

MODULUS AND CONJUGATE OF A COMPLEX NUMBER

Modulus of a complex number $z = a + ib$, denoted by $|z|$, is defined as $|z| = \sqrt{a^2 + b^2}$.

Conjugate of a complex number $z = a + ib$ denoted by $\bar{z}$, is defined as $\bar{z} = a - ib$.

PROPERTIES OF MODULUS AND CONJUGATE OF A COMPLEX NUMBER

(i) $|z_1z_2| = |z_1|\,|z_2|$

(ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, provided $|z_2| \neq 0$

(iii) $\overline{(z_1 \cdot z_2)} = (\bar{z}_1)\,(\bar{z}_2)$

(iv) $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

(v) $\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$

(vi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $(\bar{z}_2 \neq 0)$

MULTIPLICATIVE INVERSE

Let $z = a + ib$ be a non-zero complex number. Then,

$$\frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2} \Rightarrow z^{-1} = \frac{\bar{z}}{|z|^2}$$

ARGAND PLANE

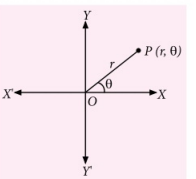
The plane in which we represent a complex number geometrically is known as the complex plane or Argand plane or Gaussian plane.

To represent $z = a + ib$ geometrically we take two mutually perpendicular straight lines $X'OX$ and $Y'OY$. Now, plot a point whose a and b coordinates are respectively the real and imaginary parts of z. This point $P(a, b)$ represents the complex number $z = a + ib$.

That is why x-axis is known as the real axis and y-axis, as the imaginary axis.

POLAR FORM

Let the complex number $z = a + ib$ be represented by the point $P(a, b)$ in the complex plane.



Let $\angle XOP = \theta$ and $|OP| = r > 0$.

Then, $P(r, \theta)$ is called the polar coordinates of P.

Clearly, we have $a = r\cos\theta$, $b = r\sin\theta$

$\therefore \quad z = r(\cos\theta + i\sin\theta)$

This is called the polar form or trigonometric form or modulus-amplitude form of z.

Here, $r = \sqrt{a^2 + b^2} = |z|$ and θ is called the argument (or amplitude) of z, written as $\arg(z)$ or $\text{amp}(z)$.

The value of θ such that $-\pi < \theta \leq \pi$ is called the principal argument of z.

PROPERTIES OF ARGUMENT OF COMPLEX NUMBERS

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$ $(k = 0 \text{ or } 1 \text{ or } -1)$

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi$ $(k = 0 \text{ or } 1 \text{ or } -1)$

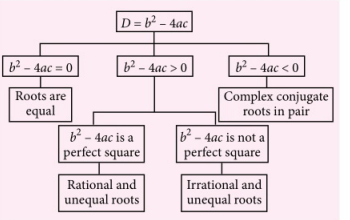
(iii) $\arg(z^n) = n \arg(z) + 2k\pi \quad (k = 0 \text{ or } 1 \text{ or } -1)$

(iv) $\arg(\overline{z}) = -\arg(z)$

QUADRATIC EQUATIONS

Any equation containing a variable of highest degree 2 is known as quadratic equation. *e.g.* $ax^2 + bx + c = 0$, then the solutions of the given equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{4ac - b^2}\,i}{2a}$$



LINEAR INEQUALITIES

DEFINITION

A statement involving variable(s) and the sign of inequality *i.e.*, >, <, ≥ or ≤ is called an inequation or an inequality.

	Inequalities	Definition	Examples
1.	Numerical	Inequalities which do not involve variables are called numerical inequalities.	$5 < 8, 5 > 2$
2.	Literal	Inequalities which involve variables are called literal inequalities.	$x > 5, y \geq 9,$ $x - y < 0$
3.	Linear	An equation of the form $ax + b < 0$, or $ax + b \geq 0$, or $ax + by > 0$ etc. are known as linear inequations.	$x + 2 < 0,$ $2x + 3y > 5$
	Inequalities	**Definition**	**Examples**
4.	Quadratic	An equation of the form $ax^2 + bx + c < 0$, or $ax^2 + bx + c \leq 0$, or $ax^2 + bx + c > 0$, or $ax^2 + bx + c \geq 0$ is known as a quadratic inequation.	$x^2 + 5x + 3 > 0$
5.	Strict	Inequalities involving the symbols '>' or '<' are called strict inequalities.	$x + y > 5,$ $y < 0$
6.	Slack	Inequalities involving the symbols '≥' or '≤' are called slack inequalities.	$5x + y \geq 1,$ $y \leq -1$

RULES FOR SOLVING INEQUALITIES

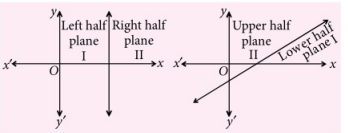
Rule 1 : Adding the same number or expression to each side of an inequation does not change the inequality.

Rule 2 : Subtracting the same number or expression from each side of an inequation does not change the inequality.

Rule 3 : Multiplying (or dividing) each side of an inequation by the same positive number does not change the inequality.

Rule 4 : Multiplying (or dividing) each side of an inequation by the same negative number reverses the inequality.

GRAPHICAL SOLUTION OF LINEAR INEQUALITIES IN TWO VARIABLES



The region containing all the solutions of an inequality is called the solution region.

(i) In order to identify the half plane represented by an inequality, it is just sufficient to take any point (a, b) not on line and check whether it satisfies the inequality or not.

If it satisfies, then the inequality represents the half plane and shade the region which contains the point, otherwise the inequality represents the half plane which does not contain the point within it. For the convenience, the point $(0, 0)$ is preferred.

(ii) If an inequality is of type $ax + by \geq c$ or $ax + by \leq c$, then the point on the line $ax + by = c$ are also included in the solution region. So draw a dark line in the solution region.

(iii) If an inequality is of the type $ax + by > c$ or $ax + by < c$, then the points on the line $ax + by = c$ are not to be included in the solution region. So draw a broken or dotted line in the solution region.

VERY SHORT ANSWER TYPE (1 MARK)

1. Find the multiplicative inverse of complex number $4 - 3i$.

2. Express $\frac{i}{(1+i)}$ in the form $(a + ib)$.

3. Solve the equation $x^2 + 3 = 0$.

4. Find the multiplicative inverse of $(2+\sqrt{3}\,i)^2$.

5. Solve the linear inequation $3x - 7 > x + 1$.

SHORT ANSWER TYPE (4 MARKS)

6. Reduce $\left(\frac{1}{1-2i}+\frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$ to the standard form.

7. Represent $(-1-\sqrt{3}i)$ in the polar form.

8. Solve the following equation by factorization method $x^2 - \sqrt{2}i\,x + 12 = 0$.

9. If $a+ib=\frac{(x+i)^2}{2x^2+1}$, then prove that $a^2+b^2=\frac{(x^2+1)^2}{(2x^2+1)^2}$

10. Solve $2x - 3 < x + 2 \leq 3x + 5$, $x \in R$. Draw the graph of the solution set.

LONG ANSWER TYPE (6 MARKS)

11. Find the modulus and argument of the following complex number and convert it in polar form : $\frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$.

12. Solve : $x^2 - (7 - i)\,x + (18 - i) = 0$.

13. Solve : $3x^2 + 8ix + 3 = 0$.

14. Solve : $\frac{2}{|x-3|} > 5$, $x \in R$.

15. A small manufacturing firm produces two types of gadgets A and B, which are first processed in the foundry and then sent to another machine for finishing. The number of man-hours of labour required in each shop for the production of each unit of A and of B and the number of man-hours for the firm available per week are as follows:

	Foundry	Machine-shop
Man-hours for 1 unit of gadget A	10	5
Man-hours for 1 unit of gadget B	6	4
Firm's capacity per week in man-hours	1000	600

Formulate it in the form of linear inequations. Draw the graph showing the solution of all these inequations.

SOLUTIONS

1. Multiplicative inverse of $4 - 3i$

$$= \frac{1}{4-3i} = \frac{1}{4-3i} \times \frac{4+3i}{4+3i} = \frac{4+3i}{(4)^2-(3i)^2}$$

$$= \frac{4+3i}{16-9i^2} = \frac{4+3i}{16+9} = \frac{1}{25}(4+3i)$$

2. We have :

$$\frac{i}{(1+i)} = \frac{i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{(i-i^2)}{(1-i^2)} = \frac{i-(-1)}{1-(-1)} = \frac{i+1}{2} = \left(\frac{1}{2}+\frac{1}{2}i\right).$$

3. Here, $x^2 + 3 = 0 \Rightarrow x^2 = -3$

$\Rightarrow \quad x = \pm\sqrt{-3} = \pm\sqrt{3}\, i$

4. Let $z = (2+\sqrt{3}\, i)^2$

$\Rightarrow \quad z = 4 + 3i^2 + 4\sqrt{3}i = 1 + 4\sqrt{3}i$

$\therefore \quad \frac{1}{z} = \frac{1}{1+4\sqrt{3}\, i} = \frac{1-4\sqrt{3}i}{(1+4\sqrt{3}i)(1-4\sqrt{3}i)} = \frac{1-4\sqrt{3}i}{1+48}$

$$= \frac{1}{49} - \frac{4\sqrt{3}i}{49}$$

5. Given, $3x - 7 > x + 1$

$\Rightarrow \quad 2x > 8 \quad \Rightarrow \quad x > 4 \quad \Rightarrow \quad x \in (4, \infty)$

6. We have,

$$\frac{1}{(1-2i)} = \frac{1}{(1-2i)} \times \frac{1+2i}{1+2i} = \frac{(1+2i)}{(1^2-4i^2)}$$

$$= \frac{(1+2i)}{5} = \left(\frac{1}{5}+\frac{2}{5}i\right)$$

$$\frac{3}{(1+i)} = \frac{3}{(1+i)} \times \frac{(1-i)}{(1-i)} = \frac{3(1-i)}{(1^2-i^2)} = \left(\frac{3}{2}-\frac{3}{2}i\right)$$

$$\frac{(3+4i)}{(2-4i)} = \frac{(3+4i)}{(2-4i)} \times \frac{(2+4i)}{(2+4i)} = \frac{(6-16)+(12+8)i}{(4-16i^2)}$$

$$= \frac{-10+20i}{20} = \left(\frac{-1}{2}+i\right)$$

$\therefore$ Given expression

$$= \left[\left(\frac{1}{5}+\frac{2}{5}i\right)+\left(\frac{3}{2}-\frac{3}{2}i\right)\right]\left(-\frac{1}{2}+i\right)$$

$$= \left[\left(\frac{1}{5}+\frac{3}{2}\right)+\left(\frac{2}{5}-\frac{3}{2}\right)i\right]\left(\frac{-1+2i}{2}\right)$$

$$= \left(\frac{17}{10}-\frac{11}{10}i\right)\left(\frac{-1+2i}{2}\right) = \frac{(17-11i)}{10} \times \frac{(-1+2i)}{2}$$

$$= \frac{(-17+22)+(34+11)i}{20} = \frac{(5+45i)}{20}$$

$$= \left(\frac{5}{20}+\frac{45}{20}i\right) = \left(\frac{1}{4}+\frac{9}{4}i\right).$$

7. Let $z = (-1-\sqrt{3}i)$

Then, z lies in quadrant III.

Let $z = r(\cos\theta + i\sin\theta)$. Then,

$r\cos\theta = -1$ ... (i)

and $r\sin\theta = -\sqrt{3}$ ... (ii)

On squaring and adding (i) and (ii), we get

$r^2 = 4 \Rightarrow r = 2$

$\therefore \quad \cos\theta = \frac{-1}{2}$ and $\sin\theta = \frac{-\sqrt{3}}{2}$

$\therefore \quad \tan\theta = \sqrt{3} = \tan\frac{\pi}{3}$

Since, $(-1, -\sqrt{3}i)$ lies in quadrant III

$\therefore$ Principal value of θ is $-\pi + \frac{\pi}{3} = \frac{-2\pi}{3}$

Hence, $z = 2\left\{\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right\}$

which is the required polar form.

8. We have, $x^2 - \sqrt{2}i\, x + 12 = 0$

$\Rightarrow \quad x^2 - 3\sqrt{2}\, i\, x + 2\sqrt{2}\, i\, x - 12i^2 = 0$

$\Rightarrow \quad x(x - 3\sqrt{2}\, i) + 2\sqrt{2}\, i(x - 3\sqrt{2}\, i) = 0$

$\Rightarrow \quad (x - 3\sqrt{2}\, i)(x + 2\sqrt{2}\, i) = 0$

$\Rightarrow \quad x - 3\sqrt{2}\, i = 0 \quad \text{or} \quad x + 2\sqrt{2}\, i = 0$

$\Rightarrow \quad x = 3\sqrt{2}\, i, \quad \text{or} \quad x = -2\sqrt{2}\, i$

9. We have, $a + ib = \frac{(x+i)^2}{2x^2+1}$ ...(i)

Taking conjugate in (i) on both sides, we get

$$\Rightarrow \quad \overline{a+ib} = \left[\frac{(x+i)^2}{2x^2+1}\right]$$

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$$\Rightarrow \overline{a+ib}=\frac{\overline{(x+i)^2}}{2x^2+1}$$

$$\Rightarrow a-ib=\frac{(x-i)^2}{2x^2+1} \quad ...(ii)$$

Multiplying (i) and (ii), we get

$$(a+ib)(a-ib)=\frac{(x+i)^2(x-i)^2}{(2x^2+1)(2x^2+1)}$$

$$\Rightarrow a^2+b^2=\frac{(x^2+1)^2}{(2x^2+1)^2}$$ Hence proved.

10. We have $2x-3<x+2\le 3x+5$

$\Rightarrow 2x-3<x+2$ and $x+2\le 3x+5$

Now, $2x-3<x+2 \Rightarrow x-3<2 \Rightarrow x<5$ ...(i)

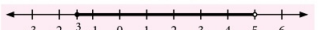
And, $x+2\le 3x+5$

$\Rightarrow 3x+5\ge x+2 \Rightarrow 2x+5\ge 2$

$\Rightarrow 2x\ge -3 \Rightarrow x\ge\frac{-3}{2} \Rightarrow \frac{-3}{2}\le x$ ...(ii)

From (i) and (ii), we get

$\Rightarrow \frac{-3}{2}\le x<5$ *i.e.* $x\in\left[\frac{-3}{2},5\right)$



11. Let $z=\frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$.

$$\Rightarrow z=\frac{i-1}{\frac{1}{2}+i\frac{\sqrt{3}}{2}}=\frac{2(-1+i)}{1+i\sqrt{3}}\times\frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$

$$\Rightarrow z=\frac{2\{(-1+\sqrt{3})+i(1+\sqrt{3})\}}{1+3}$$

$$=\left(\frac{\sqrt{3}-1}{2}\right)+i\left(\frac{\sqrt{3}+1}{2}\right)$$

Hence, $|z|=\sqrt{\left(\frac{\sqrt{3}-1}{2}\right)^2+\left(\frac{\sqrt{3}+1}{2}\right)^2}$

$$=\sqrt{\frac{(\sqrt{3}-1)^2+(\sqrt{3}+1)^2}{4}}=\sqrt{\frac{2(3+1)}{4}}=\sqrt{2}$$

Let α be the acute angle given by

$$\tan\alpha=\left|\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right|$$

Then, $\tan\alpha=\left|\frac{\frac{\sqrt{3}+1}{2}}{\frac{\sqrt{3}-1}{2}}\right|=\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$

$$=\frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{6}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{6}}=\tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right)=\tan\left(\frac{5\pi}{12}\right)$$

$\Rightarrow \alpha=\frac{5\pi}{12}$

Clearly, $\mathrm{Re}(z)=\frac{\sqrt{3}-1}{2}>0$ and

$\mathrm{Im}(z)=\frac{\sqrt{3}+1}{2}>0$. So, z lies in the first quadrant.

$\therefore \quad \arg(z)=\frac{5\pi}{12}$

Hence, the polar form of z is

$$\sqrt{2}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$$

12. Comparing the given equation with general form $ax^2+bx+c=0$, we get

$a=1, b=-(7-i)$ and $c=18-i$

$$\alpha=\frac{(7-i)+\sqrt{(7-i)^2-4(18-i)}}{2},$$

$$\beta=\frac{(7-i)-\sqrt{(7-i)^2-4(18-i)}}{2}$$

$$\Rightarrow \alpha=\frac{(7-i)+\sqrt{-24-10i}}{2},$$

$$\beta=\frac{(7-i)-\sqrt{-24-10i}}{2} \quad ...(i)$$

Now we shall first find $\sqrt{-24-10i}$

Let, $x+iy=\sqrt{-24-10i}$. Then,

$(x+iy)^2=-24-10i \Rightarrow (x^2-y^2)+2ixy=-24-10i$

On comparing, we get

$x^2-y^2=-24$ ...(ii)

and $2xy=-10$ ...(iii)

Now, $(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2$

$\Rightarrow (x^2+y^2)^2=576+100=676$

$\Rightarrow x^2+y^2=26$ ...(iv)

Solving (ii) and (iv), we get $x = \pm 1$ and $y = \pm 5$.
From (iii), we find that xy is negative.
$\therefore \quad x = 1, y = -5 \quad \text{or} \quad x = -1, y = 5.$
$\therefore \quad x + iy = 1 - 5i \quad \text{or} \quad -1 + 5i$
$\Rightarrow \quad \sqrt{-24 - 10i} = \pm (1 - 5i)$
Substituting either of these values in (i), we get

$$\alpha = \frac{7 - i + 1 - 5i}{2} = 4 - 3i \text{ and}$$

$$\beta = \frac{(7 - i) - (1 - 5i)}{2} = 3 + 2i$$

Hence, the roots of the given equation are $4 - 3i$ and $3 + 2i$.

13. The given equation is of the form
$ax^2 + bx + c = 0$, where $a = 3$, $b = 8i$ and $c = 3$.
$\therefore D = (b^2 - 4ac) = \{(8i)^2 - 4 \times 3 \times 3\}$
$= (-64 - 36) = -100 < 0$
So, the given equation has complex roots.
These roots are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8i \pm \sqrt{(8i)^2 - 4 \times 3 \times 3}}{2 \times 3}$$

$$= \frac{-8i \pm \sqrt{-100}}{6}$$

So, the roots are $\left(\frac{-8i + 10i}{6}\right)$ and $\left(\frac{-8i - 10i}{6}\right)$,

i.e. $\frac{i}{3}$ and $-3i$. $\therefore$ Solution set $= \left\{\frac{i}{3}, -3i\right\}$.

14. We have, $\frac{2}{|x-3|} > 5 \Rightarrow 2 > 5|x - 3|$

$\Rightarrow \quad |x - 3| < \frac{2}{5}$

Now, either $x - 3 > 0$ or $x - 3 < 0$.
Case I : When $x - 3 > 0$
In this case, $x - 3 > 0 \Rightarrow x > 3$
Also, $|x - 3| = x - 3$
$\therefore$ The given inequation becomes

$$x - 3 < \frac{2}{5} \Rightarrow x < \frac{2}{5} + 3 \Rightarrow x < \frac{17}{5}$$

$\therefore \quad 3 < x < \frac{17}{5}$.

Case II : When $x - 3 < 0$.
In this case, $x < 3$
Also, $|x - 3| = -(x - 3) = (3 - x)$.
$\therefore$ The given inequation becomes

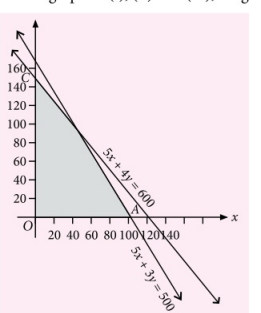
$$3 - x < \frac{2}{5} \Rightarrow -x < \frac{2}{5} - 3$$

$$\Rightarrow -x < \frac{-13}{5} \Rightarrow x > \frac{13}{5} \quad \therefore \quad \frac{13}{5} < x < 3$$

$\therefore$ Solution is $\left(\frac{13}{5} < x < 3\right)$ or $\left(3 < x < \frac{17}{5}\right)$.

$$x \in \left]\frac{13}{5}, 3\right[\cup \left]3, \frac{17}{5}\right[$$

15. Suppose that the firm manufactures x units of A and y units of B.
$\therefore$ Number of man-hours for A and B in the foundry-shop $= 10x + 6y$ and,
number of man-hours for A and B in the machine-shop $= 5x + 4y$.
$\therefore \quad 10x + 6y \le 1000$ and $5x + 4y \le 600$.
Clearly, $x \ge 0$ and $y \ge 0$
[$\because$ number of units cannot be negative].
Converting the given inequations into equations, we have
$10x + 6y = 1000 \Rightarrow 5x + 3y = 500 \quad \ldots(i)$
$5x + 4y = 600 \quad \ldots(ii)$
and $x = 0, y = 0 \quad \ldots(iii)$
Draw the graph of (i), (ii) and (iii), we get



So, the shaded region together with its boundary represents the solution set of the given inequations.

■■

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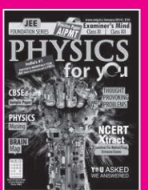

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